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# Globally nilpotent differential operators and the square Ising model

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## Abstract

We recall various multiple integrals with one parameter, related to the isotropic square Ising model, and corresponding, respectively, to the  $n$ -particle contributions of the magnetic susceptibility, to the (lattice) form factors, to the two-point correlation functions and to their  $\lambda$ -extensions. The univariate analytic functions defined by these integrals are holonomic and even  $G$ -functions: they satisfy Fuchsian linear differential equations with polynomial coefficients and have some arithmetic properties. We recall the explicit forms, found in previous work, of these Fuchsian equations, as well as their Russian-doll and direct sum structures. These differential operators are selected Fuchsian linear differential operators, and their remarkable properties have a deep geometrical origin: they are all globally nilpotent, or, sometimes, even have zero  $p$ -curvature. We also display miscellaneous examples of globally nilpotent operators emerging from enumerative combinatorics problems for which no integral representation is yet known. Focusing on the factorized parts of all these operators, we find out that the global nilpotence of the factors (resp.  $p$ -curvature nullity) corresponds to a set of selected structures of algebraic geometry: elliptic curves, modular curves, curves of genus five, six, . . . , and even a remarkable weight-1 modular form emerging in the three-particle contribution  $\chi^{(3)}$  of the magnetic susceptibility of the square Ising model. Noticeably, this associated weight-1 modular form is also seen in the factors of the differential operator for another  $n$ -fold integral of the Ising class,  $\Phi_H^{(3)}$ , for the staircase polygons counting, and in Apéry's study of  $\zeta(3)$ .  $G$ -functions naturally occur as solutions of globally nilpotent operators. In the case where we do not have  $G$ -functions, but Hamburger functions (one irregular

singularity at 0 or  $\infty$ ) that correspond to the confluence of singularities in the scaling limit, the  $p$ -curvature is also found to verify new structures associated with simple deformations of the nilpotent property.

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## 1. Introduction

Generating large series expansions of physical quantities that are quite often defined as  $n$ -fold integrals is the bread and butter of lattice statistical mechanics, enumerative combinatorics, and more generally theoretical physics. The  $n$ -fold integrals considered in theoretical physics are integrals of some more or less simple *algebraic integrands*: they are therefore holonomic [1, 2]. We actually found explicitly [3, 4] the (highly non-trivial) Fuchsian linear ODEs satisfied by the first  $n$ -particle contribution  $\chi^{(n)}$  of the magnetic susceptibility of the isotropic square Ising model for  $n = 3, 4$  (and  $n = 5$  modulo a prime [5]). Mathematicians used to say of such  $n$ -fold integrals of algebraic integrands that they are ‘*derived from geometry*’ (DFG), which means that they can be interpreted as periods<sup>6</sup> of some algebraic variety [8] closely related to the algebraic integrand<sup>7</sup>. Leaving behind all the cohomology that can be done on these algebraic varieties and other mixed Hodge structures [10], let us just recall that such DFG quantities are remarkably selected and structured. Considering the roots of the indicial polynomials of these ODEs (the critical exponents), one finds out that *all* the critical exponents of *all* the singularities of the corresponding *minimal-order* Fuchsian linear ODEs are necessarily *rational numbers* [1, 11]. Coming back to series expansions, these DFG  $n$ -fold integrals necessarily correspond to, not only Gevrey series, but *convergent series* and, often, to *arithmetic Gevrey series* [11–16]. There is a notion of order of Gevrey series: order zero corresponding to  $G$ -functions [11, 17–20], and order one to ‘Hamburger’ [21] functions<sup>8</sup>, that is to say ODEs with an *irregular singularity only*<sup>9</sup> at  $\infty$ . These  $n$ -fold integrals, corresponding to Fuchsian linear ODEs, are thus necessarily  $G$ -functions i.e. solutions of linear differential equations with arithmetic properties [11, 18, 19]. In a series of papers [24–29], the Chudnovskys underlined the crucial role of this fundamental class of functions<sup>10</sup>. They proved in [26] that solutions of linear differential equations satisfying an *arithmetic* growth property, the  $G$ -property<sup>11</sup>, have special *geometric* properties in the sense that the corresponding minimal linear differential operators are *globally nilpotent*<sup>12</sup>. From an arithmetic, as well as effective (i.e. computational) viewpoint, globally nilpotent linear differential operators [31, 32] correspond to highly selected structures with a large number of remarkable properties. In particular, their Wronskians are  $N$ th roots of rational functions. In

<sup>6</sup> See Picard–Fuchs operators [6] and *Gauss–Manin connection* [7].

<sup>7</sup> These  $n$ -fold integrals can also be seen as the ‘diagonal’ of an algebraic expression closely linked to the algebraic integrand [9].

<sup>8</sup> In that case the integrand in these  $n$ -fold integrals is not an algebraic function anymore.

<sup>9</sup> The irregular singularity is at  $\infty$  or 0, but not 0 and  $\infty$ . Mathematicians do not like to consider direct sums of ODEs corresponding to  $G$ -functions and ‘Hamburger’ functions [21]: the sum of Gevrey series of different order is *not summable* and requires the multi-summability introduced by Écalle [22, 23].

<sup>10</sup> First introduced by Siegel [30].

<sup>11</sup> The  $G$ -property is an arithmetic growth property, on the coefficients of a solution series.

<sup>12</sup> In particular, all solutions of such globally nilpotent operators with algebraic initial conditions are  $G$ -functions.

fact, this property holds as soon as the indicial polynomials for each singularity have *integer* coefficients<sup>13</sup>. A much more selected property is that *all* the critical exponents of *all* the regular singularities of these ODEs are necessarily *rational numbers*<sup>14</sup>. Recall, however, that the rationality of critical exponents is a consequence of the DFG structure.

Unfortunately this Grothendieck's geometry viewpoint is not very well known in physics, and thus, like Monsieur Jourdain<sup>15</sup>, theoretical physicists often study series expansions (low or high temperature series expansions, generating functions, . . .), or divergent series, without knowing that they are Gevrey series (and *arithmetic Gevrey series*) and often *G*-functions [18, 19]. In particular, they take for granted the rational character of the critical exponents, or the algebraic simplicity of the Wronskians of the ODEs they encounter. Many simple remarkable results on form factors, or non-trivial identities on some well-posed hypergeometric series and other Bayley pairs [36, 37], are not sufficiently recognized as a straight consequence of the fact that *G*-functions naturally occur. Along this hypergeometric line<sup>16</sup>, the paradigm of functions that can be interpreted as periods on an algebraic variety are the *hypergeometric function*  ${}_2F_1$  (or sometimes complete elliptic integrals of the first or second kind  $K, E$ ), or, more generally, hypergeometric functions  ${}_{n+1}F_n$  for some selected arguments. Dwork conjectured [39] that globally nilpotent linear differential operators of second order, are necessarily reducible, up to a rational pullback and up to the  $N$ th root of a rational function, to the hypergeometric functions  ${}_2F_1$ . This initial conjecture was ruled out by Krammer<sup>17</sup> who provided a counterexample [40] which comes from the periods of a family of Abelian surfaces over a *Shimura curve* ('wrong' elliptic curves)  $\mathbb{P}^1 \setminus \{0, 1, 81, \infty\}$ . Later other examples that are not even associated with arithmetic Fuchsian lattices, or Shimura curves, were also found [41]. Now the conjecture is rephrased to embed hypergeometric functions  ${}_2F_1$  and such counterexamples. In this paper, we will call 'Dworkian' a globally nilpotent linear differential operator of second-order corresponding to this 'extended' conjecture.

We will try to promote a DFG Grothendieck's viewpoint using a *learn-by-example approach* which focuses on the (quite arithmetic) notion of *global nilpotence* of the linear differential operators of various holonomic quantities we already encountered in physics. Alternatively, one could also imagine to perform systematic analysis of the differential Galois group of the corresponding linear differential operators in order to obtain a deeper understanding of these operators. To some extent differential Galois group analysis and  $p$ -curvature calculations (see below) are very close. In practice, the analysis of the differential Galois group requires much more time and becomes *very difficult to perform for linear differential operators of order larger than four*, which is the vast majority of the operators we encounter in physics. In contrast,  $p$ -curvature calculations provide more partial information (only a finite amount of primes can be checked), but are easy to perform, simple, and effective.

We will consider many  $n$ -fold integrals, or generating functions, that originate from the square Ising model or from enumerative combinatorics. Some quantities are not naturally expressed as  $n$ -fold integrals of an algebraic integrand: the discovery of the global nilpotence of the corresponding (minimal order Fuchsian linear differential) operator has to be seen as a strong indication that they are DFG (can be expressed as  $n$ -fold integrals of an

<sup>13</sup> Generically the indicial polynomials of Fuchsian ODEs, with coefficients in  $\mathbb{C}[x]$ , do not have integer coefficients. In our examples originating from a lattice problem we do have integer coefficients (see appendix C).

<sup>14</sup> In this arithmetic framework, this can be seen as a consequence of Kronecker's theorem. When the polynomial coefficients of Fuchsian ODEs have integer coefficients the critical exponents are algebraic numbers. In a globally nilpotent context the critical exponents necessarily reduce to integers modulo every prime. Algebraic integers reducing to integers modulo every prime are necessarily rational numbers (Kronecker's theorem [33–35]).

<sup>15</sup> Le Bourgeois Gentilhomme (Molière).

<sup>16</sup> Which is not a surprise for Yang–Baxter specialists, see for instance [38].

<sup>17</sup> In fact, this operator was first introduced by the Chudnovskys.

algebraic integrand). Other quantities, like noticeably the  $\chi^{(n)}$  contributions of the magnetic susceptibility of the square Ising model [3, 4], are defined as  $n$ -fold integrals of an algebraic integrand<sup>18</sup>. The integrand of the algebraic function<sup>19</sup> can be chosen continuous and single valued on the torus of integration: *they are indeed* [43] *a family of periods* (in other words DFG).

In this last case, our purpose is not to give another proof of this global nilpotence<sup>20</sup>, but to see *how* these linear differential operators manage to be globally nilpotent. The corresponding minimal order differential operators can be of a quite high order (for example order 33 for  $\chi^{(5)}$ , see [5]), but are always factorized<sup>21</sup> into linear differential operators of smaller orders. Necessarily, all these factors *have to be* globally nilpotent<sup>22</sup>. The global nilpotence of order-one linear differential operators is easy to see: their Wronskians are  $N$ th roots of rational functions. The global nilpotence of order-two linear differential operators is much more interesting: are they ‘Dworkian’ operators (see before), and, more specifically, do they correspond to  ${}_2F_1$  functions, or do they correspond to counterexamples similar to Krammer’s, where Heun<sup>23</sup> functions occur, or to the more general counterexamples of Dettweiler and Reiter [41]? We will also display many globally nilpotent linear differential operators of order three, four, etc. Are they reducible to the global nilpotence of the previous order-two operators, because they are equivalent<sup>24</sup>, to symmetric squares, symmetric cubes, . . . of globally nilpotent linear differential operators of order two? Do they correspond to selected  ${}_{n+1}F_n$ ? We will see that the answers to these questions are quite non-trivial, and shed an interesting light on the very nature of the globally nilpotent operators emerging from physics.

From an ‘experimental mathematics’ viewpoint, checking the global nilpotence of linear differential operators amounts to studying these operators *modulo as many primes as possible* [47, 48], more precisely by calculating the  $p$ -curvature of these differential linear operators mod prime for different primes (see below). Along this line it is worth recalling that, in a previous paper [5], we performed massive calculations on series expansions of many  $n$ -fold integrals. These massive calculations were performed modulo various prime numbers and enabled to get many highly non-trivial exact results on these physical quantities, thus showing that modulo prime calculations are not artificial or academic: they are actually a very powerful, and efficient, tool to get highly non-trivial exact results and they are possibly the only way to get some ‘extreme’ results in physics. The  $p$ -curvature calculations performed here are a natural extension of the series and ODE mod prime calculations performed in [5]. Let us recall that the linear differential operators that annihilate our  $n$ -fold integrals, factorize in operators of much smaller order. In this paper, we will systematically calculate the  $p$ -curvature of these linear differential operators (for moderate  $p$ ) but also of each of the differential factors in their factorization (in direct sums and in products of differential operators). Except in the examples for which we do not have an  $n$ -fold integral representation of the holonomic function, these two sets of  $p$ -curvature calculations are not performed to check a global nilpotence that we

<sup>18</sup> The corresponding linear differential operators are thus holonomic [2, 42], because the integral of a holonomic  $D$ -module is necessarily holonomic.

<sup>19</sup> In the integration variables  $e^{2i\pi\phi_n}$  and in the ‘parameter’  $w$ , see (10) below.

<sup>20</sup> The  $n$ -fold integrals (over a closed  $n$ -cycle) of rational expressions are necessarily DFG. More generally,  $n$ -fold integrals (over a closed  $n$ -cycle) of rational expressions on some algebraic variety [1, 44, 45] are necessarily DFG.

<sup>21</sup> As direct sum factorizations or straight factorizations.

<sup>22</sup> The product of globally nilpotent operators is necessarily globally nilpotent. More precisely, the characteristic polynomial of the  $p$ -curvature of the product operator is the product of the characteristic polynomial of the  $p$ -curvatures of the factors (see theorem 5 in [31] or corollary 2.1.3. in [39]).

<sup>23</sup> Their ODEs are straight generalizations of hypergeometric ODEs, four singular points  $[0, 1, \alpha, \infty]$  replacing the three points  $[0, 1, \infty]$  of  ${}_2F_1$  (Heun functions generalizing  ${}_2F_1$ ).

<sup>24</sup> In the sense of the equivalence of linear differential operators [46]. We refer to this (classical) notion of equivalence of linear differential operators everywhere in this paper.

know to be a simple consequence of the integral (of an algebraic integrand) representation, but to get a *global understanding* of these ODEs *beyond the usual local analysis* (singularities, exponents, formal series expansions, see the notion of ‘accessory parameters’ [49–52] below) and get more precise detail on these operators<sup>25</sup>.

The paper is organized as follows: we first recall a few  $n$ -fold integrals and some basic facts on global nilpotence. The factorization of the linear differential operators annihilating these holonomic  $n$ -fold integrals will provide a bunch of non-trivial examples of globally nilpotent operators of growing orders. We successively consider the global nilpotence of such operators of orders two, three and four. This study will provide a deep understanding of these global nilpotence from the discovery of the *underlying structures of selected algebraic varieties* (hypergeometric functions with a Hauptmodul pullback, various modular structures, etc). We will finally show that there is clearly a generalization of global nilpotence to be discovered in the scaling limit of our lattice models. We will conclude with a systematic program of analysis of  $n$ -fold integrals in theoretical physics.

In our learn-by-examples approach the variable in the ODEs<sup>26</sup> is generally called  $x$ , except when we need to recall previous results on the Ising model, where the variables were called  $w$  or  $t$ , or when we need to introduce some change of variables.

## 2. Recalls on Fuchsianity and global nilpotence

### 2.1. Recalls on Fuchsianity for lattice problems

When a (minimal order) linear differential operator with polynomial coefficients is discovered for series expansions in lattice statistical mechanics, or enumerative combinatorics on a lattice, one always finds out that it is a Fuchsian differential operator. This *lattice* property is not true in the scaling limit (see section 10). The regularity of two singular points of the ODE, 0 and  $\infty$ , is a simple consequence of the well-known existence of various kinds of series expansions (low-temperature, high-temperature, high-field, large  $q$  expansions, . . .). The fact that the other singularities are regular may seem more mysterious at first, except if one remarks that these series with rational number coefficients, or even integer coefficients, have a finite radius of convergence, and, in fact, are  $G$ -functions (see section 7).

It is important to remark that the Fuchsian ODEs one encounters in lattice statistical mechanics, or enumerative combinatorics are *not* the most generic Fuchsian ODEs, but very *selected* ones. One inherits from their *lattice origin* the fact that their polynomial coefficients have *integer coefficients*, that all the *indicial polynomials of all the singular points have integer coefficients*, and, thus, that all the critical exponents are *algebraic numbers, but not necessarily rational ones* (see appendix C).

Let us consider such an order  $q$  Fuchsian linear operator. Denote  $x_k$  the  $n$  regular singularities, including the apparent ones and excluding the point at infinity, and  $\rho_k^{(j)}$  the local exponents corresponding to the singularity  $x_k$ . It is straightforward (using Fuchs’ relations) to write the rational coefficient in front of the  $(q - 1)$ -derivative in terms of the local exponents of the various singularities:

$$D_x^q + \sum_{k=1}^n \frac{q \cdot (q - 1)/2 - \sum_{j=1}^q \rho_k^{(j)}}{x - x_k} \cdot D_x^{q-1} + \dots, \quad D_x = \frac{d}{dx}.$$

<sup>25</sup> Like the characteristic and minimal polynomial of the  $p$ -curvature, the Jordan-block reduction of the  $p$ -curvature, hopefully in order to get some hint on the factorization or direct-sum decompositions of these operators.

<sup>26</sup> To avoid multiplying the notations, we will sometimes use the same notations for different operators when there is no possibility of confusion.

The local exponents at the regular singular point  $x_k$  are roots of an indicial equation which is a polynomial in  $\rho$  with integer coefficients,  $a_0^{(k)} + a_1^{(k)}\rho + \dots + a_{q-1}^{(k)}\rho^{q-1} + a_q^{(k)}\rho^q = 0$ , and, thus, the sum  $\sum_{j=1}^q \rho_k^{(j)} = -a_{q-1}^{(k)}/a_q^{(k)}$ , associated with the singular point  $x_k$ , is necessarily a rational number. One sees, as a consequence, that there exists an integer  $N$  such that the  $N$ th power of the Wronskian

$$W(x) = \prod_{k=1}^n (x - x_k)^{\sum_{j=1}^q \rho_k^{(j)} - q(q-1)/2} \tag{1}$$

is a rational function.

All the examples of Fuchsian ODEs displayed in this paper (see in particular (21) below) have Wronskians that are  $N$ th roots of rational functions, and, as we just saw it, this is straightforwardly inherited from the underlying lattice. Lattice statistical mechanics and enumerative combinatorics naturally provide Fuchsian ODEs with  $N$ th roots of rational function Wronskians, and algebraic numbers critical exponents.

We will show, in the following, that *an even more selected set of Fuchsian linear differential operators naturally occurs in theoretical physics, the globally nilpotent operators*: the previous, at first sight, algebraic numbers critical exponents have *necessarily* to be rational critical exponents<sup>27</sup>.

### 2.2. Recalls on global nilpotence

A linear differential homogeneous equation of order  $q$ , with polynomial coefficients in  $\mathbb{Q}[x]$ , can always be written as a first-order system of homogeneous linear differential equations:

$$Y' = A \cdot Y, \quad Y = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(q-1)} \end{bmatrix} \quad \text{or} \quad \left( \frac{d}{dx} - A \right) \cdot Y = 0, \tag{2}$$

where the entries of the matrix  $A$  are rational functions of  $x$ . Instead of studying the connection  $d/dx - A$ , one can, alternatively, consider for almost any<sup>28</sup> prime  $p$ , its  $p$ -iterate modulo  $p$ :

$$\psi_p = \left( \frac{d}{dx} - A \right)^p, \quad \text{mod } p. \tag{3}$$

This  $\psi_p$  is called the  $p$ -curvature, and *it turns out that this  $p$ -curvature for any prime number  $p$ , is a  $\mathbb{F}_p(x)$ -linear operator, so that it can be represented by a matrix whose entries are rational functions of  $x$ , rather than a linear differential operator*. The differential system  $Y' = A \cdot Y$  yields for the  $i$ th derivative of  $Y$ :

$$Y^{(i)} = A_i \cdot Y, \quad \text{with: } A_{i+1} = \frac{dA_i}{dx} + A_i \cdot A, \quad A_1 = A. \tag{4}$$

Katz shows [58] that computing the  $p$ -curvature amounts to calculating  $A_p$  modulo  $p$  from the Lie sequence (4). This can be done by performing  $p$  products of  $q \times q$  matrices whose entries are rational functions in  $\mathbb{F}_p(x)$  (i.e. rational functions with coefficients in  $\mathbb{F}_p$  where

<sup>27</sup> *Stricto sensu* the rationality of all the critical exponents of a Fuchsian ODE is *not sufficient* to have the global nilpotence property, see [53]. Global nilpotence is stronger than the rationality of all the critical exponents for our 'lattice' Fuchsian ODEs. One has conditions on the so-called *accessory parameters* [49–52].

<sup>28</sup> Almost all, here, and in the following, means for all primes except a *finite set* of primes.



$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ ). These  $p$ -curvatures were introduced in the framework of the Grothendieck conjecture, to provide ‘algebraic criteria’ for the monodromy group of (2) to be finite<sup>29</sup>.

In the case of order-one linear differential operators, Grothendieck’s conjecture was proved by Honda [53]. The fact that the exponents of the various regular singularities are *rational numbers* can be seen as a consequence of *Kronecker’s theorem which says that any algebraic number which reduces to integers modulo almost every prime is necessarily a rational number* [33–35]. The conjecture can also be proved in some particular cases: by Dwork for ordinary hypergeometric equations [39], and by Katz for Gauss–Manin differential equations (see [46, 47] for more detail on the second-order linear differential equations when one has only three regular singular points, like 0, 1,  $\infty$  for hypergeometric functions). The conjecture is still open for general second-order operators.

Rather than the Grothendieck–Katz  $p$ -curvature conjecture, one can consider various theorems by Katz [58], in particular proposition 9.3 in [58], which shows that the reductions modulo  $p$  of the Lie algebra of the differential Galois group contain the  $p$ -curvatures  $\psi_p$ .

Beyond the situation of Grothendieck’s conjecture where these  $p$ -curvatures vanish, another highly selected situation corresponds to the case where these  $p$ -curvatures are *nilpotent modulo  $p$ , for almost all primes  $p$*  (for all primes except a *finite set of primes*). In that case, the linear differential operator is called *globally nilpotent*. A globally nilpotent linear differential operator is necessarily a *Fuchsian* linear differential operator, but it has many more strong remarkable structures. For instance, all the exponents of all its various regular singularities are *rational*, but the reciprocal statement is not true: a Fuchsian linear differential operator with rational exponents is *not necessarily globally nilpotent*. Global nilpotence is a stronger structure than having regular singularities with rational exponents [53]. It is a very strong arithmetic property with a large number of remarkable consequences: for instance modulo any prime  $p$  the Fuchsian linear differential operator factorizes, and *for almost all primes, it factorizes into linear differential operators of order one*, each operator of order one having *rational solutions* modulo  $p$ . Such a property is quite well illustrated<sup>30</sup> in appendix H of [59, 60] on  $n$ -fold integrals related to Apéry’s analysis of  $\zeta(3)$ . In that case, we even have a factorization into order-one linear differential operators on the rationals  $\mathbb{Q}$  and not only modulo (almost all) primes. The fact that the solutions of these order-one linear differential operators are actually rational solutions modulo primes, is clearly reminiscent of the occurrence of Wronskians that are  $N$ th roots of rational expressions.

Global nilpotence is often said to suggest a ‘deep geometrical interpretation’, namely that the solutions of a globally nilpotent linear differential operator can be interpreted *as periods* of some (hidden, . . .) algebraic variety, suggesting more or less a *Gauss–Manin connection* [7] interpretation for these linear differential operators.

Beyond the linear differential operators associated with Apéry’s analysis of  $\zeta(3)$ , almost all examples of globally nilpotent linear differential operators correspond to hypergeometric functions, and other Katz’s rigid local systems [61], for which an interpretation of the solutions as periods of an algebraic variety plays a central role. Within the known examples, the overlap between hypergeometric functions (and their simple generalizations) and global nilpotence was so large that Dwork proposed a conjecture [39] that all the globally nilpotent linear

<sup>29</sup> The Grothendieck–Katz  $p$ -curvature conjecture is a problem on linear ordinary differential equations, related to differential Galois theory. It is a conjecture of A Grothendieck from the late 1960s, and apparently not published by him in any form; it has been publicized, reformulated and in some cases related to deformation theory proved by N Katz in a series of papers [54–57]. The question is to give an arithmetic criterion for when there is a full set of algebraic function solutions.

<sup>30</sup> Note a misprint in [59] one should read  $\ln A_i$ , instead of  $A_i$ , in the equations defining the  $A_i$  after equation (H.2) in [59, 60].



differential operators correspond to *hypergeometric functions up to simple transformations*. This conjecture was ruled out by Krammer [40]. Therefore, at the present moment, beyond the fact that it is a highly remarkable arithmetico-geometric selected property, one can say that one does not have a complete understanding of global nilpotence.

In the following, we are going to find globally nilpotent linear differential operators corresponding to various  $n$ -fold integrals that occur naturally in the case of the off-critical lattice Ising model, or corresponding to enumerative combinatorics for which no  $n$ -fold integral representation is yet known. We will also explore situations that are precious to understand namely other  $n$ -fold integrals that naturally occur in particle physics (Feynman diagrams [62]), like, for instance, some selected scaling limits.

### 2.3. Krammer's counterexample

Let us recall briefly Krammer's counterexample<sup>31</sup> to Dwork's conjecture [40, 41] which comes from the periods of a family of *Abelian surfaces* over a *Shimura curve*  $\mathbb{P}^1 \setminus \{0, 1, 81, \infty\}$ :

$$Y' = \left( \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_{81}}{x-81} \right) \cdot Y, \quad A_0 = \begin{bmatrix} 0 & 0 \\ -1/2 & -1/2 \end{bmatrix}, \quad (5)$$

$$A_1 = \begin{bmatrix} 0 & 0 \\ 4/9 & -1/2 \end{bmatrix}, \quad A_{81} = \begin{bmatrix} 0 & 1 \\ 0 & 1/2 \end{bmatrix},$$

yielding the second-order operator (on the first component of the vector  $Y$ ):

$$O_1 = D_x^2 + \frac{1}{2} \cdot \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-81} \right) \cdot D_x + \frac{x-9}{18(x-81)(x-1) \cdot x}, \quad D_x = \frac{d}{dx}, \quad (6)$$

or the second-order operator (on the second component of the vector  $Y$ ):

$$O_2 = 18x \cdot (x-1)(x-9)(x-81)^2 \cdot D_x^2 + 27(x-81) \cdot (x^3 - 123x^2 + 1491x - 729) \cdot D_x + (x^3 + 549x^2 + 13203x - 1003833). \quad (7)$$

The two linear differential operators  $O_1$  and  $O_2$  are, of course, equivalent and the squares of their Wronskians are simple rational functions. We have calculated their  $p$ -curvatures and confirmed that they are globally nilpotent.

The general solution of  $O_1$  reads in terms of Heun functions [63]:

$$\mu \cdot \text{Heun}(81, 1/2; 1/6, 1/3, 1/2, 1/2; x) + \lambda \cdot x^{1/2} \cdot \text{Heun}(81, 21; 2/3, 5/6, 3/2, 1/2; x),$$

where  $\mu$  and  $\lambda$  are two constants. The differential Galois group of (5) (or (6), (7)) is a central extension of  $SL(2, C)$ . Calculating the indicial polynomials of  $O_1$  at the various singularities, one finds the indicial polynomial  $(6r-1) \cdot (3r-1)$  for  $t = \infty$ , and  $r \cdot (2r-1)$  for the singularities  $t = 0, 1, 81$ . These Heun functions *cannot be reduced* [40] to hypergeometric functions  ${}_2F_1$  (up to multiplication and some pullback). Along this line, it is important to note that, generically, a Heun function *does not* correspond to a globally nilpotent second-order differential operator. For instance we calculated the  $p$ -curvature of a second-order operator very similar to (6):

$$D_x^2 + \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-81} \right) \cdot D_x + \frac{1}{2} \frac{81-28x}{(x-81)(x-1)x}, \quad (8)$$

<sup>31</sup> The uniformizing linear differential equation of an arithmetic Fuchsian lattice [40].

which has as solutions the Heun functions

$$\mu \cdot \text{Heun}(81, -81/2, -7/2, 4, 1/2, 1/2; x) + \lambda \cdot x^{1/2} \cdot \text{Heun}(81, -20, 9/2, -3, 3/2, 1/2; x),$$

and we found that (8) is *not* globally nilpotent.

### 3. Global nilpotence of a few $n$ -fold integrals of the Ising class

#### 3.1. Recalls of a few $n$ -fold integrals of the Ising class

The susceptibility of the Ising model can be written [64] as an infinite sum of  $n$ -fold integrals. These  $n$ -particle contributions  $\chi^{(n)}$  are given by  $(n - 1)$ -dimensional integrals [65–67] that read

$$\tilde{\chi}^{(n)}(w) = \frac{1}{n!} \cdot \left( \prod_{j=1}^{n-1} \int_0^{2\pi} \frac{d\phi_j}{2\pi} \right) \left( \prod_{j=1}^n y_j \right) \cdot R^{(n)} \cdot (G^{(n)})^2, \tag{9}$$

where<sup>32</sup>

$$G^{(n)} = \prod_{1 \leq i < j \leq n} h_{ij}, \quad h_{ij} = \frac{2 \sin((\phi_i - \phi_j)/2) \cdot \sqrt{x_i x_j}}{1 - x_i x_j} \tag{10}$$

and

$$R^{(n)} = \frac{1 + \prod_{i=1}^n x_i}{1 - \prod_{i=1}^n x_i}, \tag{11}$$

with

$$x_i = \frac{2w}{1 - 2w \cos(\phi_i) + \sqrt{(1 - 2w \cos(\phi_i))^2 - 4w^2}}, \tag{12}$$

$$y_i = \frac{2w}{\sqrt{(1 - 2w \cos(\phi_i))^2 - 4w^2}}, \quad \sum_{j=1}^n \phi_j = 0 \tag{13}$$

valid for small  $w$  and, elsewhere, by analytical continuation. We actually found [3, 4, 68, 69] the linear ODEs for some of these holonomic  $n$ -particle contributions namely  $\chi^{(3)}$ ,  $\chi^{(4)}$  and, *modulo a prime*, for  $\chi^{(5)}$ . From an arithmetic Gevrey series and  $G$ -function viewpoint it is worth noting that the series expansion of the  $\tilde{\chi}^{(n)}$ , in the variable  $w$ , are series expansions with *integer* coefficients:

$$\tilde{\chi}^{(n)}(w) = 2^n \cdot w^{n^2} \cdot (1 + 4n^2 \cdot w^2 + 2 \cdot (4n^4 + 13n^2 + 1) \cdot w^4 + \frac{8}{3} \cdot (n^2 + 4)(4n^4 + 23n^2 + 3) \cdot w^6 + \dots), \tag{14}$$

where the  $w^2$  coefficient is valid for  $n \geq 3$ , the  $w^4$  coefficient is valid for  $n \geq 5$  and the  $w^6$  coefficient is valid for  $n \geq 7$ . Note that the  $w^6$  coefficient is *always an integer*<sup>33</sup>.

In previous publications [59, 70], we also introduced some integrals of the so-called Ising class<sup>34</sup>. We considered several kinds of integral representations (one-dimensional and multidimensional) of these holonomic functions which belong to the Ising class [71]. Again we obtained the linear ODEs of these sets of integrals for the first values of  $n$ , through series

<sup>32</sup> The fermionic term  $G^{(n)}$  has several representations [66].

<sup>33</sup> It would be interesting to get much longer series expansion like (14), valid for arbitrary  $n$ , to see if these successive rational functions of  $n$  are actually functions of  $n^2$ .

<sup>34</sup> The terminology *integral of the Ising class* has been proposed by Bailey et al in [71].

expansions [59, 70]. In [59] a detailed analysis of the multiple integrals  $\Phi_H^{(n)}$  was performed. These  $n$ -fold integrals amount to removing the Fermionic factor  $G^{(n)}$  in (9), so that one introduces

$$\Phi_H^{(n)}(w) = \frac{1}{n!} \cdot \left( \prod_{j=1}^{n-1} \int_0^{2\pi} \frac{d\phi_j}{2\pi} \right) \left( \prod_{j=1}^n y_j \right) \cdot \frac{1 + \prod_{i=1}^n x_i}{1 - \prod_{i=1}^n x_i}. \tag{15}$$

Even simpler integrals (over a *single* variable) were also introduced and denoted [70] by  $\Phi_D^{(n)}$ :

$$\Phi_D^{(n)}(w) = -\frac{1}{n!} + \frac{2}{n!} \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{1}{1 - x^{n-1}(\phi) \cdot x((n-1)\phi)}, \tag{16}$$

where  $x(\phi)$  is given by (12).

### 3.2. Results on global nilpotence of these $n$ -fold integrals of the Ising class

Let us display here our results for the calculations of  $p$ -curvatures for the *minimal order* ODEs of  $n$ -fold integrals (9), (15) and (16) of the Ising class.

We have calculated (modulo the first thousand primes) the  $p$ -curvature of the order-six linear differential operator  $L_6$  occurring in  $\chi^{(3)}$  (see (22) in section 4.1 and [69]), as well as the Jordan-block reduction of the  $6 \times 6$   $p$ -curvature matrix, and found that the characteristic polynomial of the  $p$ -curvature reads  $T^6$ . This  $6 \times 6$  Jordan-block reduction can be compared with the two  $3 \times 3$  Jordan-block reductions corresponding to (the  $p$ -curvature of) an order-three operator  $Z_2 \cdot N_1$  that righdivides  $L_6$ , and another order-three operator  $Y_3$  that leftdivides  $L_6$  (see section 4.1). They read respectively (in block form):

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The explicit Jordan-block form of the  $p$ -curvature is quite reminiscent of the factorization of the operator  $L_6$  (see section 4.1) in an order-three linear differential operator, and another order-three operator, itself product of an order-two operator and an order-one operator. Note, however, that one should not extrapolate beyond simple product factorizations: the Jordan-block form of the  $p$ -curvature gives systematically a misleading prejudice of direct-sum structures that do not exist.

The global nilpotence of the order-ten Fuchsian linear differential operator [68] for  $\chi^{(4)}$  is confirmed by the calculation of the  $p$ -curvature for all the primes up to  $p \leq 809$ . The  $p$ -curvature *has been found to be nilpotent for all these primes*.

The global nilpotence of the order-five and six Fuchsian linear differential operators for  $\Phi_H^{(3)}$  and  $\Phi_H^{(4)}$  is confirmed by the calculation of the  $p$ -curvature: we have calculated the  $p$ -curvature for all the primes up to  $p \leq 809$  and it *has been found to be nilpotent for all these primes*. The characteristic polynomial of the  $p$ -curvature of the (globally nilpotent) linear differential operator of  $\Phi_H^{(3)}$  has been found to be  $T^5$  (its minimal polynomial being  $T^3$ ). The characteristic polynomial of the  $p$ -curvature of the (globally nilpotent) linear differential operator of  $\Phi_H^{(4)}$  has been found to be  $T^5$  (its minimal polynomial being  $T^4$ ). For  $\Phi_H^{(5)}$  the calculations are drastically larger, but, from a *probabilistic algorithm*, we found that the characteristic polynomial of the  $p$ -curvature of the linear differential operator of  $\Phi_H^{(5)}$  is  $T^{17}$ .

The minimal polynomial of  $p$ -curvature of  $\Phi_D^{(3)}$  and  $\Phi_D^{(4)}$  is  $T^4$ . The characteristic polynomial of the  $p$ -curvature of  $\Phi_D^{(5)}$  and  $\Phi_D^{(6)}$  is  $T^5$ . The minimal polynomial of  $p$ -curvature of  $\Phi_D^{(8)}$  is  $T^6$ . Recall that the characteristic polynomial of the  $p$ -curvature of a globally nilpotent operator of minimal order  $N$  equals  $T^N$ .

### 3.3. Other $n$ -fold integrals of the Ising class

Other  $n$ -fold integrals (corresponding to the susceptibility of a square Ising model for which a magnetic field is located only on spins on a particular diagonal of the square lattice) were introduced in [72]. For instance, for  $T < T_c$ , they read

$$\begin{aligned} \tilde{\chi}_{d-}^{(2n)}(t) &= \frac{t^{n^2}}{(n!)^2} \frac{1}{\pi^{2n}} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n} dx_k \cdot \frac{1 + t^n x_1 \cdots x_{2n}}{1 - t^n x_1 \cdots x_{2n}} \\ &\quad \times \prod_{j=1}^n \left( \frac{x_{2j-1}(1-x_{2j})(1-tx_{2j})}{x_{2j}(1-x_{2j-1})(1-tx_{2j-1})} \right)^{1/2} \\ &\quad \times \prod_{1 \leq j \leq n} \prod_{1 \leq k \leq n} (1 - tx_{2j-1}x_{2k})^{-2} \prod_{1 \leq j < k \leq n} (x_{2j-1} - x_{2k-1})^2 (x_{2j} - x_{2k})^2 \end{aligned} \quad (17)$$

and another similar formula for  $\tilde{\chi}_{d+}^{(2n+1)}(t)$ . They are holonomic functions and their corresponding Fuchsian linear differential operators were given in [72]. Again the calculations of the  $p$ -curvature of the corresponding linear differential equations of minimal order for  $\tilde{\chi}_{d+}^{(3)}$  and  $\tilde{\chi}_{d-}^{(4)}$  confirmed their global nilpotence (see section 6.1 and appendix D).

### 3.4. ODEs for two-point correlation functions and form factors

Many simple linear ODEs of various orders were obtained for the two-point<sup>35</sup> correlation functions of the square Ising model [73]. The two-point correlation functions were found to be polynomials (with rational function coefficients) of complete elliptic integrals of the first and second kinds: their global nilpotence is, thus, a straight consequence of their hypergeometric nature.

Along this correlation function line, we can recall the linear differential operators  $F_j(N)$  we obtained for the form factors<sup>36</sup>  $f_{N,N}^{(j)}$  of the (off-critical) square Ising model [74] and, in particular, their Russian-doll structure.

The linear differential operators  $F_{2n+1}(N)$ , which annihilate the form factors  $f_{N,N}^{(2n+1)}$  have a ‘Russian-doll’ structure. They are such that

$$\begin{aligned} F_1(N) &= L_2(N), \\ F_3(N) &= L_4(N) \cdot L_2(N), \\ F_5(N) &= L_6(N) \cdot L_4(N) \cdot L_2(N), \dots \end{aligned} \quad (18)$$

where the differential operators  $L_r(N)$  are of order  $r$ . The first one reads

$$L_2(N) = D_t^2 + \frac{2t-1}{(t-1)t} \cdot D_t - \frac{1}{4t} + \frac{1}{4(t-1)} - \frac{N^2}{4t^2}, \quad (19)$$

and the expressions of  $L_4(N)$ ,  $L_6(N)$ ,  $L_8(N)$  and  $L_{10}(N)$  are given in [74].

Thus we see that the linear differential operator for  $f_{N,N}^{(2n-1)}$  rightdivides the differential operator for  $f_{N,N}^{(2n+1)}$ ,  $n \leq 3$ . Similar relations occur for the  $F_{2n}(N)$  s. We conjectured [74] that this property holds for all values of  $n$ . We thus have a ‘Russian-doll’ (telescopic) structure of these successive linear differential operators.

Again, these form factors were found to be polynomial (with rational function coefficients) of complete elliptic integrals of the first and second kinds: the global nilpotence of the corresponding operators is, again, a straight consequence of their hypergeometric nature.

<sup>35</sup> Or could have been obtained for any  $N$ -point correlation functions.

<sup>36</sup> Coefficients in  $\lambda^j$  of  $C(N, N; \lambda)$ , the  $\lambda$ -extension [74] of the two-point correlation function  $C(N, N)$ .

### 3.5. Modular ODEs for lattice form factors

Along this correlation function line, it is also worth recalling the Fuchsian linear ODEs we found [74] for some  $\lambda$ -extensions  $C(N, N; \lambda)$  of two-point correlation functions of the (off-critical) lattice Ising model for selected values of the parameter  $\lambda$ . As examples of these Fuchsian linear differential operators, we found, for instance, that  $C_-(N, N; \cos(\pi/4))$ , for  $N = 0, 1, 2, \dots$ , are annihilated, respectively, by<sup>37</sup>

$$\begin{aligned} L_0^{[1/4]} &= (t-1)^2 t \cdot D_t^2 + \frac{3}{8}(t-1)(3t-2) \cdot Dt - \frac{15t}{256} + \frac{3}{32}, \\ L_1^{[1/4]} &= (t-1)^2 t \cdot D_t^2 + \frac{(t-1)(5t-2)}{8} \cdot Dt - \frac{7t}{256} + \frac{1}{16}, \\ L_2^{[1/4]} &= (t-8)(t-1)^2 t \cdot D_t^2 + \frac{7}{8} \cdot (t-1)(t^2-2t+16) \cdot Dt + \frac{209t^2}{256} - \frac{25t}{16} + \frac{1}{2}. \end{aligned} \tag{20}$$

One has homomorphisms between the linear differential operators with same parity: the  $L_n^{[1/4]}$  for  $n$  odd (resp. even) are equivalent [47]. We also have higher order ODEs like, for instance, the order-four linear differential operators [74], denoted by  $L_N^{[1/3]}$ , corresponding to  $C_-(N, N; \cos(\pi/3))$ .

We have calculated the  $p$ -curvatures of all these irreducible linear differential operators and seen that they have *zero  $p$ -curvatures*. Not surprisingly the corresponding Wronskians associated with these  $\lambda$ -extensions of two-point correlation functions are  $N$ th root of rational functions and read:

$$\begin{aligned} W(L_0^{[1/4]}) &= (1-t)^{-3/8} \cdot t^{-3/4}, & W(L_1^{[1/4]}) &= (1-t)^{-3} t^{-2}, \\ W(L_2^{[1/4]}) &= (t-8)^{-8/7} \cdot (1-t)^{-15/7} \cdot t^{-2/7}, & \dots, \\ W(L_0^{[1/3]}) &= (1-t)^{-11/3} \cdot t^{-11/3}, & W(L_1^{[1/3]}) &= t^{-14/3} \cdot (1-t)^{-8/3}, \\ W(L_2^{[1/3]}) &= (11+21t) \cdot (1-t)^{1/3} \cdot t^{-20/3}, & \dots \end{aligned} \tag{21}$$

The fact that the eighth power (instead of the square in most of the examples of this paper) of  $W(L_0^{[1/4]})$ , or the seventh power of  $W(L_2^{[1/4]})$ , is rational is in agreement with the interpretation of  $\lambda$  we gave in [74]. These Fuchsian linear ODEs actually correspond to *algebraic functions*<sup>38</sup>, and are often, explicitly, associated with *modular curves*.

### 3.6. More zero $p$ -curvatures: Joyce's Fuchsian ODEs

Finally, along this zero  $p$ -curvature line, it is also worth recalling the large set of higher order Fuchsian ODEs obtained by Joyce [75–80]. There are not so many examples of Fuchsian linear ODEs of high order in the literature. Joyce has been one of the few authors to provide such non-trivial examples. We have calculated the  $p$ -curvature of a large set of these Fuchsian linear differential operators, namely (42) in [76], (85) and (86) in [77], (5.22) of [75], (2.16) of [78], ... We found that they are more than globally nilpotent: their  $p$ -curvature is *zero for almost every prime* (for all primes except a finite set of primes). They have a basis of *algebraic solutions* that can be expressed in terms of simple Legendre-P functions and simple algebraic functions. These Fuchsian linear ODEs actually correspond to *algebraic functions*, and are often, explicitly, associated with *modular curves*.

<sup>37</sup> Note two misprints in [74] for  $L_1^{[1/4]}$  and  $L_2^{[1/4]}$ , corresponding to the  $D_t$  coefficient.

<sup>38</sup> Note that maple also solves these ODEs in terms of (algebraic) hypergeometric or Heun functions (see (35), (37)).

#### 4. Global nilpotence from the global nilpotence of the factors

##### 4.1. Revisiting the global nilpotence of $\chi^{(3)}$

The minimal linear differential operator for  $\chi^{(3)}$  is an order-seven operator  $L_7$  which can be written as the direct sum of the order-one linear differential operator for  $\chi^{(1)}$  and an order-six linear differential operator for  $2\chi^{(3)} - \chi^{(1)}$ , namely  $L_6$  which factorizes into an order-three linear differential operator  $Y_3$ , an order-two linear differential operator  $Z_2$  and an order-one linear differential operator  $N_1$ :

$$L_6 = Y_3 \cdot Z_2 \cdot N_1. \tag{22}$$

The explicit expressions of  $Y_3$  and  $Z_2$  are given in appendix A of [69].

We have the following, almost obvious remark<sup>39</sup>, that *the product of globally nilpotent operators is necessarily globally nilpotent*. The global nilpotence of the order-one linear differential operator  $N_1$  is obvious. Furthermore, we found the remarkable result that the solutions of  $Y_3$  are quadratic expressions of the complete elliptic integrals of the first and second kinds  $K$  and  $E$  (see appendix B of [69]). From a differential algebra viewpoint this amounts to saying that  $Y_3$  is *equivalent to the symmetric square* of the second-order linear differential operator  $L_E$  corresponding to  $E$  (see [81]). Since hypergeometric functions correspond to globally nilpotent operators,  $Y_3$  is therefore globally nilpotent.

The global nilpotence of the linear differential operator for  $\chi^{(3)}$  thus reduces to the global nilpotence of the second-order linear differential operator  $Z_2$ . The linear differential operator  $Z_2$ , is an example of globally nilpotent operator which does not straightforwardly reduce to hypergeometric functions up to change of variables (pullback) and multiplications.

In fact, a simple right-multiplication of  $Z_2$  by  $h(w) = 1/(1 + 4w)/(1 - 4w)^2$ , enables one to get rid of the singularity  $w = -1/4$ . Instead of the solutions  $F(w)$  of  $Z_2$ , this amounts to considering the second-order linear differential operator  $\tilde{Z}_2$  with solutions  $F(w)/h(w)$ . Denoting  $D_w = d/dw$ , this linear differential operator reads

$$\tilde{Z}_2 = \frac{1}{h(w)} \cdot Z_2 \cdot h(w) = q_2 \cdot q_{app} \cdot D_w^2 + q_1 \cdot D_w + 24w \cdot q_0,$$

with

$$\begin{aligned} q_2 &= w \cdot (1 - w) (1 - 4w) (1 + 2w)(1 + 3w + 4w^2), \\ q_1 &= 1 - 2w + w^2 - 216w^3 - 336w^4 + 1656w^5 + 1040w^6 - 2560w^7 - 6400w^8 - 6144w^9, \\ q_0 &= 1 - 7w - 4w^2 - 47w^3 + 36w^4 + 280w^5 + 160w^6 + 256w^7, \\ q_{app} &= 1 - 3w - 18w^2 + 104w^3 + 96w^4. \end{aligned} \tag{23}$$

The polynomial  $q_{app}$  corresponds to *apparent* singularities. All the other singularities are regular singularities *remarkably with integer exponents and all yielding logarithmic behaviours*. This can simply be seen from their corresponding indicial polynomials and the formal series around these singularities. The differential Galois group of  $\tilde{Z}_2$  is  $SL(2, C)$ , a consequence of the existence of logs in the formal solutions around singular points, together with a Wronskian being rational, and the operator being irreducible [82].

Calculating the indicial polynomials at the various singularities, one finds the following critical exponents:

$$\rho = 0, 0, \quad \text{for} \quad w = 0, 1/4, \frac{-3 \pm i\sqrt{7}}{8},$$

<sup>39</sup> If one takes as a definition of global nilpotence the factorization modulo (almost all) primes of the operator in order-one operators with rational function solutions modulo primes (see [31] and lemma 0.6.2 in [39]).

$$\begin{aligned} \rho = 0, 2, & \quad \text{for} \quad w = 1, -1/2, \\ \rho = -1, -2 & \quad \text{for} \quad w = \infty. \end{aligned} \tag{24}$$

If one wants to see the solution as hypergeometric functions up to a pullback, the change of variables to be done must try to ‘wrap’ all these seven singularities onto only three ones: 0, 1,  $\infty$ . Naively one can think of wrapping the singularities according to the previous roots of indicial polynomials (24), namely  $w = 0, w = 1/4, (-3 \pm i\sqrt{7})/8 \rightarrow 0, w = 1, w = -1/2, \rightarrow 1/4, w = \infty \rightarrow \infty$ , however, since the critical exponents are all integers, these seven singularities have, in fact, to be considered *on the same footing*. This will be discussed in more detail in section 5.1, where the explicit solution of  $Z_2$  will be given *in terms of a modular form of weight one*.

#### 4.2. Revisiting the global nilpotence of $\chi^{(4)}$

For the order-ten linear differential operator of  $\chi^{(4)}$  we have similar calculations. The order-ten linear differential operator  $L_{10}$  of  $\chi^{(4)}$  is the direct sum of the order-two linear differential operator for  $\chi^{(2)}$  and of an order-eight linear differential operator which factorizes [68, 69] into an order-four operator and four order-one operators (see (F.4) in [68]):

$$\mathcal{L}_8 = M_2 \cdot L_{25} \cdot L_{12} \cdot L_3 \cdot L_0. \tag{25}$$

The global nilpotence of the order-one operators  $L_{25}, L_{12}, L_3$  and  $L_0$  is a simple consequence of the fact that all these operators are of the form  $D_x + R'/R$ , where  $R$  denotes a rational function, or the square of a rational function (simply related to the Wronskian of these operators), and  $R'$  its first derivative with respect to  $x$ . These  $R$  functions read respectively for  $L_{25}, L_0, L_{12}$  and  $L_3$ :

$$\begin{aligned} R_{25} &= \frac{(5x + 7)(4 - x)^{7/2}}{(14 - 12x + 9x^2 - 5x^3) \cdot x^2 (1 - x)^2}, & R_0 &= 1, \\ R_{12} &= \frac{(14 - 12x + 9x^2 - 5x^3) \cdot x}{(2 + 3x^2 + x^3)(1 - x)^{7/2}}, & R_3 &= \frac{2 + 3x^2 + x^3}{(1 - x)^2 \cdot x^{3/2}}. \end{aligned} \tag{26}$$

The order-two linear differential operators  $N_i, i = 1, \dots, 9$ , sketched in [69] and which happen in other factorizations of  $L_{10}$ , are all equivalent to the order-two linear differential operator [69]  $N_0$

$$N_0 = D_x^2 - \frac{1+x}{1-x} \cdot \frac{D_x}{x} + \frac{1}{4x} \cdot \frac{1}{1-x}, \tag{27}$$

having hypergeometric solutions (corresponding to  $\chi^{(2)}$ ):

$$\chi^{(2)} = x^2 \cdot {}_2F_1([3/2, 5/2], [3], x). \tag{28}$$

Most of the ‘complexity’ of  $\chi^{(4)}$  is thus ‘encapsulated’ in the order-four linear differential operators  $M_2$  (or equivalently  $M_1$  of [69]).

In a remark (in appendix B of [69], p 27), we mentioned the fact that the solutions of  $M_2$  can be expressed as linear combinations of products of complete elliptic integrals. One can make this statement more precise. Let us introduce the *symmetric cube* of the linear operator [69]  $N_0$  (associated with  $\chi^{(2)}$ ), the order-four linear differential operators  $M_2$  (or equivalently  $M_1$ ) is *equivalent* to the symmetric cube  $\text{Sym}^3(N_0)$ . There exist two order-three interwinners  $I_1$  and  $I_2$  such that:

$$I_1 \cdot \text{Sym}^3(N_0) = M_2 \cdot I_2. \tag{29}$$

In other words, the solutions of  $M_2$  are cubic (homogeneous) polynomials [69] of the two solutions of  $N_0$  (hypergeometric functions). Therefore  $M_2$  is *globally nilpotent*, and



consequently the order-ten linear differential operator for  $\chi^{(4)}$  is also globally nilpotent. One sees that, paradoxically, the global nilpotence of the linear differential operator for  $\chi^{(4)}$  is much simpler to understand than the global nilpotence of the linear differential operator for  $\chi^{(3)}$ , which is a consequence of the, after first sight, subtle global nilpotence of  $Z_2$ .

#### 4.3. Revisiting the global nilpotence of $\Phi_H^{(3)}$

In order to revisit the global nilpotence of  $\Phi_H^{(3)}$  (see (15)), let us study the factorization of the corresponding order-five Fuchsian linear operator given in [59]. This order-five Fuchsian linear operator factorizes into an order-three and an order-two Fuchsian linear operators:

$$L_{\Phi_H^{(3)}} = M_3 \cdot M_2, \tag{30}$$

where the second-order operator  $M_2$  is given in appendix B.2. The Wronskian of the order-two operator  $M_2$  is a simple rational function. The indicial polynomials for  $M_2$  yield integer critical exponents, when the formal series solutions show logarithms for all the singularities, the second-order operator  $M_2$  being irreducible. Consequently [82], the differential Galois group of  $M_2$  is  $SL(2, C)$ .

We have calculated the  $p$ -curvature of  $M_3$  and  $M_2$  and found that their characteristic polynomial equals their minimal polynomials, being respectively  $T^3$  and  $T^2$  for almost all primes. These two linear differential operators are thus globally nilpotent.

More remarkably, we actually found that  $M_2$  is equivalent<sup>40</sup> to  $Z_2$  of  $\chi^{(3)}$  (see (41)). We also found that  $M_3$  is equivalent to  $Y_3$  of  $\chi^{(3)}$ , or equivalently  $\text{Sym}^2(Q_E)$ , the symmetric square of the linear differential operator  $Q_E$  corresponding to the complete elliptic integral  $E(4x)$ :

$$Q_E = D_x^2 + \frac{D_x}{x} + \frac{16}{(1 - 4x)(1 + 4x)}. \tag{31}$$

Therefore  $\chi^{(3)}$  and  $\Phi_H^{(3)}$  have extremely close structures.

As usual the order-two intertwinners  $H_1$  and  $H_2$  (or  $H'_1$  and  $H'_2$ ) in the equivalence of  $M_3$  and  $Y_3$

$$M_3 \cdot H_1 = H_2 \cdot \text{Sym}^2(Q_E), \quad H'_1 \cdot M_3 = \text{Sym}^2(Q_E) \cdot H'_2$$

are themselves equivalent and, again, there exist two order-one intertwinners  $K_1$  and  $K_2$  (resp.  $K'_1$  and  $K'_2$ ) such that  $K_1 \cdot H_1 = H_2 \cdot K_2$ , and  $K'_1 \cdot H'_1 = H'_2 \cdot K'_2$ , which are again equivalent. One thus has a ‘tower’ of equivalent differential operators. It is important to note that the ‘intertwinning’ operators in such a ‘tower’ of equivalences *are not necessarily globally nilpotent!*

#### 4.4. Revisiting the global nilpotence of $\Phi_H^{(4)}$

The global nilpotence of  $\Phi_H^{(4)}$  can be understood from the factorization of the corresponding order-six Fuchsian linear differential operator given in [59]. It factorizes as follows:

$$L_{\Phi_H^{(4)}} = M_4 \cdot P_1 \cdot Q_1, \tag{32}$$

where the order-four operator  $M_4$  is given in appendix B.3 and the two order-one operators  $P_1$  and  $Q_1$  read

$$P_1 = D_x + \frac{1}{2} \frac{d}{dx} \ln((x - 4)(x - 1)^2 \cdot x^2),$$

$$Q_1 = D_x + \frac{1}{2} \frac{d}{dx} \ln((x - 1)^2 \cdot x).$$

<sup>40</sup> Up to the change  $x = 4w$ .

The global nilpotence of  $P_1$  and  $Q_1$  corresponds to the fact that the square of the Wronskians of these first-order operators are *simple rational functions*.

The order-four operator  $M_4$  is irreducible but *it is actually equivalent to*  $\text{Sym}^3(L_E)$ , the symmetric cube of the linear differential operator  $L_E$  having  $E(x^{1/2})$  (complete elliptic integral) as solution:

$$L_E = D_x^2 + \frac{D_x}{x} + \frac{1}{4} \frac{1}{(1-x)x}. \tag{33}$$

The global nilpotence of  $\Phi_H^{(4)}$  is clearly a straight consequence of this last equivalence with a symmetric cube of a hypergeometric operator. Comparing the factors (and their equivalence) for  $\Phi_H^{(4)}$  one finds out, after the remarkable identification of structure between  $\Phi_H^{(3)}$  and  $\chi^{(3)}$ , that one has exactly the same identification of structures for  $\Phi_H^{(4)}$  and  $\chi^{(4)}$ . Again, slightly surprisingly, the global nilpotence of  $\Phi_H^{(4)}$  (or  $\chi^{(4)}$ ) is straightforward to understand compared to the global nilpotence of  $\Phi_H^{(3)}$  (or  $\chi^{(3)}$ ) which amounts to understanding the more subtle global nilpotence of the second-order operator  $Z_2$ .

#### 4.5. Revisiting the global nilpotence of the $\Phi_D^{(n)}$ 's

Similar (detailed factorizations) calculations can be performed for  $\Phi_D^{(3)}$ ,  $\Phi_D^{(4)}$ ,  $\Phi_D^{(5)}$  and  $\Phi_D^{(6)}$  defined by (16). They are displayed in appendix E. Again one finds out that the global nilpotence of many of the factors occurring in the factorizations can be explained by the occurrence of complete elliptic integrals of the first or second kind. The example of  $\Phi_D^{(6)}$  in appendix E corresponds to a much more interesting situation, also occurring with  $\Phi_D^{(5)}$ , where we contemplate in the factorization of the operator for  $\Phi_D^{(6)}$ , a remarkable combination of a globally nilpotent order-two operator associated with complete elliptic integrals of the first or second kind, *together with* a second-order operator of *zero p-curvature* associated with highly non-trivial (genus five) algebraic curve (with a group of quaternion as its differential Galois group). Let us illustrate a similar structure focusing on the global nilpotence of  $\Phi_D^{(5)}$ .

#### 4.6. Revisiting the global nilpotence of $\Phi_D^{(5)}$

The linear differential operator for  $\Phi_D^{(5)}$  is a Fuchsian linear differential operator of order five which factorizes in three different ways ( $N_4, N_3, N_2, N_1$  are second-order linear differential operators,  $L_1$  and  $L_2$  first order):

$$L_{\Phi_D^{(5)}} = L_1 \cdot N_2 \cdot N_1 = N_3 \cdot L_2 \cdot N_1 = N_3 \cdot N_4 \cdot D_x. \tag{34}$$

This means that it factorizes into the direct sum of two order-two linear differential operators and the operator  $D_x$ :

$$L_{\Phi_D^{(5)}} = M_4 \oplus D_x, \quad \text{with } M_4 = N_2 \cdot N_1,$$

where the second-order operator  $N_1$  is given in appendix B.4. The square of the Wronskian of  $N_1$  is a simple rational function.

The  $p$ -curvature of  $N_1$  is equal to zero for any prime  $\geq 2$ , therefore, modulo Grothendieck's conjecture<sup>41</sup>, it admits a basis of algebraic solutions. These algebraic solutions can be written as hypergeometric functions up to a pullback by a rational function and multiplication by an  $N$ th root of a rational function. They are of the form [83, 84]:

$$R_1(x)^{1/4} \cdot H[R_2(x)], \quad \text{where } P(H[R_2(x)], x) = 0, \tag{35}$$

<sup>41</sup> Namely that zero  $p$ -curvatures yield [14, 54–56] a basis of algebraic solutions for the linear differential operator.

where  $R_1(x)$  and  $R_2(x)$  are rational functions,  $H$  is an algebraic hypergeometric function and  $P(y, x) = 0$  is an algebraic curve of degree 4 and of genus 6:

$$(1 - x - 3x^2 + 4x^3)^2(1 + x)^2(1 + 8x + 20x^2 + 15x^3 + 4x^4) \times (1 + 2x - 4x^2)(1 - 3x + x^2)(1 + 2x)(1 - x) \cdot y^4 - 2(1 + x)(1 - x - 3x^2 + 4x^3) \cdot g_1 \cdot y^2 + (1 + 3x + 3x^2 - 4x^3)^2(1 + x - 6x^2 - x^3 + x^4)^2 = 0, \tag{36}$$

where

$$g_1 = -3 - 24x - 20x^2 + 255x^3 + 484x^4 - 800x^5 - 1729x^6 + 1296x^7 + 2236x^8 - 1035x^9 - 1004x^{10} + 232x^{11} + 160x^{12}.$$

Note that the vanishing of the  $y^4$  coefficient in (36) corresponds to singularities of  $\Phi_D^{(5)}$ .

This genus 6 algebraic curve should not be confused<sup>42</sup> with genus 3 curves that will be mentioned at the end of section appendix F.1.

Using the maple command `kovaciccols` one can write one of the solutions in terms of hypergeometric functions as follows:

$$(R_{\Phi_D^{(5)}}(x))^{1/12} \cdot {}_2F_1([-1/4, 1/4], [1/2], \mathcal{M}), \tag{37}$$

where the argument in the hypergeometric function reads

$$\mathcal{M} = \frac{\mathcal{N}}{\mathcal{D}}, \quad \text{where} \tag{38}$$

$$\mathcal{N} = (x - 1)(1 + 2x)(1 - 3x + x^2)(1 + 2x - 4x^2) \times (1 + 8x + 20x^2 + 15x^3 + 4x^4)(1 + 3x + 3x^2 - 4x^3)^2 \times (1 + x - 6x^2 - x^3 + x^4)^2,$$

$$\mathcal{D} = 8(1 + x)(1 - x - 3x^2 + 4x^3) \cdot d_{10}^2,$$

$$d_{10} = 1 + 8x + 8x^2 - 73x^3 - 148x^4 + 144x^5 + 325x^6 - 152x^7 - 168x^8 + 39x^9 + 28x^{10}. \tag{39}$$

and the rational function  $R_{\Phi_D^{(5)}}(x)$  reads

$$R_{\Phi_D^{(5)}}(x) = -\frac{n}{g_1^2 \cdot \mathcal{N}}, \quad \text{where} \tag{40}$$

$$n = 16 \cdot (x + 1)^2(1 - x - 3x^2 + 4x^3)^2 \cdot d_{10}^4.$$

The linear differential operator  $N_2$  (or equivalently  $N_3$ ) has solutions in terms of the complete elliptic integrals  $E$  and  $K$ . The  $p$ -curvatures of the linear differential operator  $N_2$  (resp.  $N_3$ ) correspond to a globally nilpotent operator.

#### 4.7. Towards a geometric interpretation of the $\Phi_D^{(n)}$

The DFG (global nilpotence) structure corresponds to the fact that a holonomic function has an interpretation as a period of some algebraic variety. Along this line, it is worth noting that in the case of these  $\Phi_D^{(n)}$ , some closed exact expressions for these integrals  $\Phi_D^{(n)}$  can be obtained which give explicit examples of such an interpretation as a period. Actually the simple integrals  $\Phi_D^{(n)}$  can all be expressed as sums of complete elliptic integrals of the third kind, where the characteristic  $y = y(w)$  corresponds to some selected, and highly non-trivial, algebraic curves (genus 0, 3, 10, ...). The results are displayed in appendix F.1.

<sup>42</sup> They can, however, correspond to highly non-trivial relations expressing such genus 6 algebraic solutions as linear combinations of complete elliptic integrals of the third kind with a ‘characteristic’ (first argument of the complete elliptic integral) associated with a genus 3 curve.

### 5. The second-order operator $Z_2$ and weight-1 modular forms

The previous section was dedicated to understanding the global nilpotence of linear differential operators of quite high orders from the global nilpotence of their factors of smaller orders. To sum up all the previous results, the global nilpotence of the factors fall in three categories:

- firstly, a straightforward global nilpotence of order-one factors (which amounts to saying that their Wronskian is an  $N$ th root of a rational function);
- secondly, global nilpotence straightforwardly associated with complete elliptic integrals of the first (or second) kind, or equivalently  ${}_2F_1$  hypergeometric functions, or global nilpotence corresponding to symmetric powers of the previous second-order hypergeometric operators, or to zero curvature operators with their solutions being a basis of algebraic functions (often modular curves, such algebraic functions can be written as pullbacks of  ${}_2F_1$  hypergeometric functions);
- finally, a set of operators of order two, three, . . . that do not have a basis of algebraic solutions and that we have not been able to immediately reduce to  ${}_2F_1$  hypergeometric functions, or products of hypergeometric functions.

The second-order linear differential operator  $Z_2$ , occurring in the factorization of the linear operator for  $\chi^{(3)}$  (or equivalently  $M_2$  for  $\Phi_H^{(3)}$ ), is a perfect illustration of this last situation. The order two and three operators occurring in the operator factorization for the three-choice polygons perimeter generating function (see below) are other examples that do not immediately fit in a  ${}_2F_1$  ‘Dworkian’ framework.

#### 5.1. Reducing $Z_2$ to Heun functions

We found that the  $Z_2$  of  $\chi^{(3)}$ , and its equivalent for  $\Phi_H^{(3)}$ , namely  $M_2$  in (30) and (B.3), are homomorphic<sup>43</sup> ( $U_1$  and  $U_2$  are two order-one intertwiners):

$$Z_2 \cdot U_1 = U_2 \cdot M_2, \quad U_1 = \frac{(1+2x)(1-x)}{(1-2x)(1+4x)(1-4x)} \cdot u_1, \tag{41}$$

$$u_1 = x \cdot (1-4x)(1+3x+4x^2) \cdot D_x + (1+x+12x^2+48x^3),$$

On the other hand, let us introduce the order-two Heunian operator whose solution is Heun(8/9, 2/3, 1, 1, 1, 1;  $t$ ):

$$\mathcal{H} = D_t^2 + \left( \frac{1}{t} + \frac{1}{t-1} + \frac{9}{9t-8} \right) \cdot D_t + 3 \frac{3t-2}{(9t-8)(t-1)t}, \tag{42}$$

a simple change of variable:

$$t = \frac{-8x}{(1-4x)(1-x)} \tag{43}$$

transforms (42) into the order-two linear differential operator:

$$\mathcal{H}_x = D_x^2 + \frac{1-10x+19x^2-92x^3+12x^4+224x^5-64x^6}{(1+3x+4x^2)(1-2x)(1+2x)(1-4x)(1-x)} \cdot D_x + 6 \frac{(1+7x+4x^2)(1-2x)^2}{(1+3x+4x^2)(1-4x)^2(1-x)^2 \cdot x}. \tag{44}$$

This operator (44) is just the conjugate of  $M_2$  by the multiplication by a simple polynomial function  $h(x) = (1-x)(1-4x)^2$ :

$$h(x) \cdot M_2 = \mathcal{H}_x \cdot h(x). \tag{45}$$

<sup>43</sup> Up to the change  $x = 4w$  in  $M_2$ .

Together with (41), this means that the second-order linear differential operator  $Z_2$ , corresponding to  $\chi^{(3)}$ , reduces to a Heun operator given by (42), to be compared with Krammer’s [40] counterexample (6) to Dwork’s conjecture.

One then gets the (selected) solution for  $Z_2$  in terms of Heun functions ( $w$  replaced here by  $x$ , and  $t$  is given by (43)):

$$r(x) \cdot ((1 - 9x)(1 - 4x)(1 - x) \cdot Hg(t) + 8 \cdot (1 + 3x + 4x^2) \cdot x \cdot Hg'(t)),$$

$$\text{where } r(x) = \frac{(1 + 2x)^2}{(1 - x)^2(1 + 4x)(1 - 4x)^2} \quad \text{and}$$

$$Hg'(t) = \frac{dHg(t)}{dt}, \quad Hg(t) = \text{Heun}(8/9, 2/3, 1, 1, 1, 1, t). \quad (46)$$

5.2. Reducing  $Z_2$  to weight-1 modular forms

Recall [85] that the *fundamental weight-1 modular form*<sup>44</sup>  $h_N$  for the modular group  $\Gamma_0(N)$  for  $N = 6$ , can be expressed as a simple Heun function,  $\text{Heun}(9/8, 3/4, 1, 1, 1, 1, -t/8)$ , or as a hypergeometric function:

$$\frac{2\sqrt{3}}{((t + 6)^3(t^3 + 18t^2 + 84t + 24)^3)^{1/12}} \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right]; [1]; 1728 \frac{(t + 9)^2(t + 8)^3 t}{(t + 6)^3(t^3 + 18t^2 + 84t + 24)^3}\right), \quad (47)$$

and that (47) is solution of the order-two linear differential operator (obtained from (42) by  $t \rightarrow -t/9$ ):

$$D_t^2 + \left(\frac{1}{t + 8} + \frac{1}{t} + \frac{1}{t + 9}\right) \cdot D_t + \frac{t + 6}{(t + 8)(t + 9) \cdot t}. \quad (48)$$

Therefore, after some changes of variables, one can see the (selected) solution of  $Z_2$  as a hypergeometric function (up to a pullback) corresponding to *weight-1 modular form*<sup>45</sup> ( $h_6$  in [85]). To sum-up  $\mathcal{H}_x$ , given by (44), has the following solution:

$$\mathcal{S} = (\Omega \cdot \mathcal{M}_x)^{1/12} \times {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right]; [1]; \mathcal{M}_x\right), \quad \text{where}$$

$$\Omega = \frac{1}{1728} \frac{(1 - 4x)^6(1 - x)^6}{x \cdot (1 + 3x + 4x^2)^2(1 + 2x)^6},$$

$$\mathcal{M}_x = 1728 \frac{x \cdot (1 + 3x + 4x^2)^2(1 + 2x)^6(1 - 4x)^6(1 - x)^6}{(1 + 7x + 4x^2)^3 \cdot P^3},$$

$$P = 1 + 237x + 1455x^2 + 4183x^3 + 5820x^4 + 3792x^5 + 64x^6. \quad (49)$$

Recalling (41) and (45), the solution of the operator  $Z_2$  in terms of hypergeometric functions then corresponds to the action of the intertwiner  $U_1$  on the solution  $\mathcal{S}/h(x)$  of  $M_2$ . The global nilpotence of  $Z_2$  can now be understood from this hypergeometric function (up to a modular invariant pullback) structure.

<sup>44</sup> The modular form  $h_6$  is also combinatorially significant: the perimeter generating function of the three-dimensional staircase polygons [86] can be expressed in terms of  $h_6$  (see section 8.1). The modular form  $h_6$  also occurs in Apéry’s study of  $\zeta(3)$  (see appendix A).

<sup>45</sup> The simplest weight-1 modular form is  ${}_2F_1([1/12, 5/12], [1], \hat{J}) = 12^{1/4} \eta(\tau)^2 \hat{J}^{-1/12}$ , where  $\hat{J}$  is the Hauptmodul,  $\eta$  is the Dedekind eta function and  $\tau$  is the ratio of periods (see (4.6) in [87]). It can also be expressed as a linear combination of  ${}_2F_1([1/12, 5/12], [1/2], 1 - \hat{J})$  and  ${}_2F_1([7/12, 11/12], [3/2], 1 - \hat{J}) \cdot (1 - \hat{J})^{1/2}$ .

5.3. Atkin’s modular functions

The modulus  $\mathcal{M}_x$  in the argument of  ${}_2F_1$  actually corresponds [85] to the (genus 0) modular curve which amounts to multiplying, or dividing, the ratio of the two periods of the elliptic curve by 6:

$$\Phi_6(j, j') = \Phi_6(j', j) = 0 \tag{50}$$

obtained from the elimination of  $z$  between

$$j = j_6(z) = \frac{(z + 6)^3(z^3 + 18z^2 + 84z + 24)^3}{z(z + 9)^2(z + 8)^3},$$

$$j = j_6\left(\frac{2^3 \cdot 3^2}{z}\right) = \frac{(z + 12)^3(z^3 + 252z^2 + 3888z + 15552)^3}{z^6(z + 8)^2(z + 9)^3}.$$
(51)

Actually, similar to (43), if one introduces

$$z = \frac{72x}{(1 - x)(1 - 4x)} \quad \text{and} \quad \mathcal{M}_z = \frac{12^3}{j_6(z)} \tag{52}$$

one finds immediately that  $\mathcal{M}_x$  in (49) is *nothing but the Hauptmodul*  $\mathcal{M}_z$ .

The singularities of the linear ODE of  $\chi^{(3)}$  (or  $\Phi_H^{(3)}$ ) correspond to the singularities of  $j(z)$ :

$$z(z + 9)(z + 8) = 5184 \frac{x \cdot (1 + 3x + 4x^2)(1 + 2x)^2}{(1 - x)^3(1 - 4x)^3}. \tag{53}$$

Similarly the ‘Atkin-dual’ [88] change of variables (see 5.3)

$$z' = \frac{72}{z} = \frac{(1 - x)(1 - 4x)}{x}, \tag{54}$$

gives

$$z'(z' + 9)(z' + 8) = \frac{(1 - x)(1 - 4x)(1 + 2x)^2(1 + 3x + 4x^2)}{x^3}. \tag{55}$$

Recalling the *modular Atkin-polynomial*<sup>46</sup> [88] for (50) (see 5.3):

$$z \cdot (z + 9)^2(z + 8)^3 \cdot j - (z + 6)^3(z^3 + 18z^2 + 84z + 24)^3, \tag{56}$$

one finds out that the singularities of the linear ODE of  $\chi^{(3)}$  are obtained from this *modular Atkin-polynomial* [88] (56) together with the covering (43) (or (52)).

These results strongly suggest that *all the singularities, and the associated polynomials with integer coefficients we obtained in previous papers (from some involved Landau singularities analysis [70, 59]), should have an interpretation as singularities of an (absolute) Klein modular invariant  $j(N \cdot \tau)$ , or equivalently of the modular Atkin-polynomials [88], for higher values of  $N$ , when rewritten in the  $w$  variable.*

The difficult part, here, corresponds to find the well-suited covering (43), or (52), ‘wrapping’ the seven singularities in  $\tilde{Z}_2$  onto the three singularities  $0, 1, \infty$  of the hypergeometric function  ${}_2F_1$ . Noticeably, the well-suited covering (43) (or (52)) *does not* correspond to a partition according to the critical exponents (24) (see section 4.1), but to the following selected partition:

$$\begin{aligned} w = 0, & & w = \infty & \rightarrow & 0, \\ w = -1/2, & & \frac{-3 \pm i\sqrt{7}}{8} & \rightarrow & 1, \\ w = 1, & & w = 1/4, & \rightarrow & \infty. \end{aligned} \tag{57}$$

<sup>46</sup> In the case of modular Atkin-polynomial of degree one in  $j$ , like (56), one sees immediately the rational parametrization (51). The introduction of the modular Atkin-polynomial becomes necessary when one does not have a rational parametrization like (51) anymore, because the genus of the modular curves is no longer zero.

To find [89] this covering (57) among the  $\sum_{p=2}^6 \binom{7}{p} \cdot (2^p - 2) = 3^7 - 3 \cdot (2^7 - 2) - 3 = 1806$  possible ones, one had to see the seven singularities on the *same footing*.

### 6. From ${}_2F_1$ with a pullback to ${}_3F_2$ with a pullback

#### 6.1. Linear differential equation for $\tilde{\chi}_d^{(3)}(t)$

For  $\chi_d^{(3)}$  (see (17)), we choose  $x = t^{1/2}$  as our independent variable. We find that the linear differential operator for  $\tilde{\chi}_d^{(3)}(x)$  is of order six, and has the *direct sum* decomposition

$$\mathcal{L}_6^{(3)} = L_1^{(3)} \oplus L_2^{(3)} \oplus L_3^{(3)}, \tag{58}$$

with

$$\begin{aligned} L_1^{(3)} &= Dx + \frac{d}{dx} \ln(x - 1), \\ L_2^{(3)} &= Dx^2 + 2 \frac{(1 + 2x)}{(1 + x)(x - 1)} \cdot Dx + \frac{1 + 2x}{(1 + x)(x - 1)x}, \end{aligned} \tag{59}$$

$$\begin{aligned} L_3^{(3)} &= Dx^3 + \frac{3(8x^6 + 36x^5 + 63x^4 + 62x^3 + 21x^2 - 6x - 4)}{2(x + 2)(1 + 2x)(1 + x)(x - 1)(1 + x + x^2)x} \cdot Dx^2 \\ &\quad + \frac{n_1(x)}{(x + 2)(1 + 2x)(1 + x)^2(1 - x)^2(1 + x + x^2)x^2} \cdot Dx \\ &\quad - \frac{n_0(x)}{(x + 2)(1 + 2x)(1 - x)^3(1 + x + x^2)(1 + x)^2x^2}, \end{aligned} \tag{60}$$

where

$$\begin{aligned} n_0(x) &= 2x^8 + 8x^7 - 7x^6 - 13x^5 - 58x^4 - 88x^3 - 52x^2 - 13x + 5, \\ n_1(x) &= 14x^8 + 71x^7 + 146x^6 + 170x^5 + 38x^4 - 112x^3 - 94x^2 - 19x + 2. \end{aligned}$$

The linear differential operator of order two,  $L_2^{(3)}$  is equivalent to the second-order operator corresponding to the complete elliptic integral  $E(x)$ :

$$\mathcal{L}_E = D_x^2 + \frac{D_x}{x} + \frac{1}{(1 - x)(1 + x)}. \tag{61}$$

Consequently, this order-two linear differential operator  $L_2^{(3)}$  is globally nilpotent: actually we have calculated its  $p$ -curvature and found that the corresponding characteristic polynomial (or minimal polynomial) reads  $T^2$ .

Similarly, the order-three linear differential operator  $L_3^{(3)}$  is globally nilpotent: we have calculated its  $p$ -curvature and found that the corresponding characteristic polynomial (or minimal polynomial) reads  $T^3$ .

#### 6.2. The solution of $L_3^{(3)}$ as a ${}_3F_2$ with a Hauptmodul pullback

The order-three linear differential operator  $L_3^{(3)}$  corresponds to a generalization of the weight-1 modular form operators which have been discovered in section 5. Actually, introducing the rational function

$$\rho(x) = \frac{(1 + 2x)(x + 2)}{(1 - x)(1 + x + x^2)}, \tag{62}$$



one can find that the three solutions of  $L_3^{(3)}$  are two MeijerG functions

$$\begin{aligned} \rho(x) \cdot \text{MeijerG}([[-1/2, 1/3, 2/3], []], [[0, 0], [0]], Q), \\ \rho(x) \cdot \text{MeijerG}([[-1/2, 1/3, 2/3], []], [[0, 0, 0], []], -Q), \end{aligned} \tag{63}$$

and a  ${}_3F_2$  hypergeometric function<sup>47</sup>:

$$\rho(x) \cdot {}_3F_2([1/3, 2/3, 3/2], [1, 1]; Q), \tag{64}$$

where  $Q$  is nothing but the Hauptmodul (the reciprocal of Klein’s invariant) with the (elliptic lambda function)  $x$  changed into  $-x$ :

$$Q = \frac{27}{4} \frac{(1+x)^2 x^2}{(x^2+x+1)^3} = \frac{1}{J(-x)} = \frac{12^3}{j(-x)}. \tag{65}$$

With (64) we have a straight generalization of the previous weight-1 modular form where the expression for  $h_6$  was given as a  ${}_2F_1$  hypergeometric function with a pullback corresponding to the Hauptmodul (52). One has here a  ${}_3F_2$  hypergeometric function<sup>48</sup> with a pullback corresponding to the Hauptmodul (65). If one seeks for a modular form interpretation of the solutions of  $L_3^{(3)}$  and, in particular, for (64), one would like to see  $L_3^{(3)}$  as equivalent to the symmetric square of a  ${}_2F_1$  operator with the same Hauptmodul pullback (65). In this respect, it is worth recalling the Clausen identity<sup>49</sup>:

$${}_2F_1([a, b], [a+b+1/2]; z)^2 = {}_3F_2([2a, a+b, 2b], [a+b+1/2, 2a+2b]; z), \tag{66}$$

and in particular

$${}_2F_1([1/6, 1/3], [1]; z)^2 = {}_3F_2([1/3, 2/3, 1/2], [1, 1]; z), \tag{67}$$

which is one of the four classes<sup>50</sup> of  ${}_3F_2$  found by Ramanujan [24] that are squares of  ${}_2F_1$  representations of complete elliptic integrals. The symmetric square of the second-order operator

$$W_1 = D_x^2 + \frac{1}{2} \cdot \frac{(2x+1)(x^2+x+2)}{x(1+x)(x^2+x+1)} \cdot D_x - \frac{3}{2(x^2+x+1)^2},$$

annihilates the left-hand side of (67) where  $z$  is taken to be equal to the ‘Hauptmodul’ (65). We actually found out that this symmetric square is equivalent to  $L_3^{(3)}$  with two simple order-one intertwiners  $V_1$  and  $V_2$ :

$$V_2 \cdot \text{Sym}^2(W_1) = L_3^{(3)} \cdot V_1, \quad \text{where} \quad V_1 = \frac{x \cdot (1+x)}{(1-x)^2} \cdot D_x + \frac{1}{2} \cdot \frac{2x^2+5x+2}{(x^2+x+1)(1-x)}. \tag{68}$$

Equivalently, we could have used the following identity<sup>51</sup> to rewrite (64):

$$\begin{aligned} {}_3F_2([1/3, 2/3, 3/2], [1, 1]; Q) &= {}_2F_1([1/6, 1/3], [1]; Q)^2 \\ &+ \frac{2Q}{9} \cdot {}_2F_1([1/6, 1/3], [1]; Q) \cdot {}_2F_1([7/6, 4/3], [2]; Q) \end{aligned}$$

where  ${}_2F_1([7/6, 4/3], [2]; Q) = 18 \frac{d}{dQ} {}_2F_1([1/6, 1/3], [1]; Q).$  (69)

This generalization of the Clausen identity concludes our modular form interpretation [91, 92] of the third-order operator  $L_3^{(3)}$ .

<sup>47</sup> Reminiscent of, for instance,  ${}_2F_1([1/3, 2/3], [1]; Q)$  in (23) of [90].

<sup>48</sup> Note that this hypergeometric function is not a well-poised hypergeometric function.

<sup>49</sup> For such selected arguments of the hypergeometric function we also have the highly remarkable Gauss–Kummer quadratic relation:  ${}_2F_1([a, b], [a+b+1/2]; 4z(1-z)) = {}_2F_1([2a, 2b], [a+b+1/2]; z)$ .

<sup>50</sup> Four arithmetic triangle subgroups, commensurable with the full modular group, yielding four hypergeometric representation of periods of elliptic curves [28].

<sup>51</sup> Other rewritings involve the Heun functions  $\text{Heun}(10, 23/9, 1/6, 1/3, 3/2, 1, 1 - Q)$  and  $\text{Heun}(10, -35/18, -1/3, -1/6, 1/2, 1, 1 - Q) \cdot (1 - Q)^{-1/2}$ .

**7. Global nilpotence without integral representation: three-choice polygons, directed compact percolation, vicious walkers, . . .**

Let us also give here three more examples of global nilpotence that *do not correspond to  $n$ -fold integrals*, emerging from the theory of the Ising model, or to  $n$ -fold integrals of the Ising class, but to enumerative combinatorics. Their global nilpotence, or the fact that they could be DFG cannot be immediately seen from a representation as an  $n$ -fold integral of an algebraic expression<sup>52</sup>. Of course one can always imagine to see this global nilpotence as a consequence of a detailed analysis of the corresponding series expansions, showing explicitly that they are probably arithmetic Gevrey series and in fact  $G$ -functions.

Along this line, it is worth noting that, for instance, the series expansion of the  $\tilde{\chi}^{(n)}$  in the variable  $w$  are series expansions with *integer* coefficients (see (14)). The fact that these series of holonomic functions are series with a finite radius of convergence and with integer coefficients, gives us a strong prejudice that we are studying  $G$ -functions. For the various examples of enumerative combinatorics displayed in this section, we have a similar strong prejudice in favour of arithmetic Gevrey series:  $G$ -functions. Instead of such an arithmetic approach, we prefer to consider directly the global nilpotence, performing  $p$ -curvature calculations modulo large set of primes, since this approach is *simple, algorithmic and effective*.

*7.1. Global nilpotence without (known) integral representation: three-choice polygons*

An order-eight Fuchsian linear differential operator, was found for the perimeter generating function of the three-choice<sup>53</sup> polygons [95]. It is the direct sum of two order-one operators and an order-six linear differential operator  $M_6$ .

This order-six linear differential operator  $M_6$  is the product of an order-three, an order-two and an order-one linear differential operator, respectively  $M_3$ ,  $M_2$  and  $M_1$ :

$$M_6 = M_3 \cdot M_2 \cdot M_1, \quad \text{where} \quad M_1 = D_x - \frac{d}{dx} \ln(1 - x), \tag{70}$$

$$M_2 = D_x^2 + \frac{P_{11}}{(x - 1) \cdot P_{12} \cdot x} \cdot D_x + \frac{P_{21}}{4 \cdot P_{22}},$$

where the two order-three and order-two operators are given in (B.1) and have, beyond apparent singularities, singularities at  $x = 0$  and  $x = 1$  and at the roots of the quadratic polynomial  $16 + 4x + 7x^2$ . Let us consider the order-two linear differential operator  $M_2$ . The indicial polynomials of these singularities read

$$\begin{aligned} x = 0, & \rightarrow r \cdot (r + 2), & 16 + 4x + 7x^2 = 0, & \rightarrow r \cdot (r - 1), \\ x = 1, & \rightarrow (2r + 3)^2, & x = \infty, & \rightarrow (2r + 1) \cdot (2r - 1). \end{aligned}$$

All the formal series around these singularities have logarithms, except  $x = \infty$  which has only *square roots singularities* ( $x^{-1/2}$ ). Keeping in mind that the Wronskian of  $M_2$  is a rational function, one deduces [83] that the differential Galois group of  $M_2$  is  $SL(2, C)$ .

We have calculated the  $p$ -curvatures of the two linear differential operators of order three and two of (B.1). We found that  $M_3$  (resp.  $M_2$ ) is globally nilpotent, the characteristic polynomial of its  $p$ -curvature being  $T^3$  (resp.  $T^2$ ) for almost all primes.

<sup>52</sup> Recall that the integral of a closed differential  $n$ -form over a closed  $n$ -cycle is said to come from algebraic geometry. In our case we have an algebraic  $n$ -form, so we automatically have a closed  $n$ -form and our integration on hypercubes and not cycles, can be reduced to cycles because the global nilpotence property [93] remains stable [94] by the required extensions (desingularization, relative de Rham cohomology, etc) [Y André private communication].

<sup>53</sup> Note that there is a factor 4 in the definition of  $x$  between the ODE displayed here and that given in [95].

7.2. Global nilpotence without (known) integral representation: directed compact percolation

Generating functions, associated with the mean cluster size and length for the *directed compact percolation* problem, satisfy various linear ODEs that are displayed in [96]. For instance, equation (12) of [96], when rewritten in a homogeneous way, corresponds to the order-three linear differential operator which is the direct sum  $L_1 \oplus L_2$ , of an order-one operator  $L_1$  and an order-two operator  $L_2$ :

$$\begin{aligned}
 L_1 &= D_t + \frac{d}{dt} \ln \left( \frac{(1-2t)(1-t)^3}{2t^4 + 2t^3 - 6t^2 + 4t - 1} \right), \\
 L_2 &= D_t^2 + 2 \frac{12t^4 - 12t^3 - 11t^2 + 8t - 1}{(1-2t)(t-1)t(1+4t-4t^2)} \cdot D_t - 6 \frac{4t^4 - 12t^3 + 7t^2 - 2t + 1}{(1-2t)^2(1+4t-4t^2)(t-1)^2t}.
 \end{aligned}
 \tag{71}$$

Similarly equations (16), (18) and (21) of [96], when rewritten in a homogeneous way correspond to the order-three linear differential operator which is the direct sum  $L'_1 \oplus L'_2$ , of an order-one operator  $L'_1$  and an order-two operator  $L'_2$ . For instance, for equation (16) of [96] we have

$$\begin{aligned}
 L'_1 &= D_t + \frac{d}{dt} \ln \left( \frac{t^3}{4t^2 + 8t - 1} \right), \\
 L'_2 &= D_t^2 + \frac{24t^4 - 72t^3 + 46t^2 + 8t - 7}{(1-2t)(t-1)(1+4t-4t^2)t} \cdot D_t + \frac{8t^4 - 24t^3 + 2t^2 + 28t - 9}{(t-1)(1-2t)(1+4t-4t^2)t^2}.
 \end{aligned}
 \tag{72}$$

All these order-two operators ( $L_2, L'_2, \dots$ ) can be seen to be equivalent and are globally nilpotent.

7.3. Global nilpotence without (known) integral representation: vicious walkers

Another example of enumerative combinatorics corresponds to the vicious walkers and friendly walkers generating functions [97]. Equation (4.34) of [97], when rewritten in a homogeneous way, corresponds to the order-three linear differential operator which is the direct sum  $L_1 \oplus L_2$ , of an order-one operator  $L_1$  and an order-two operator  $L_2$ :

$$\begin{aligned}
 L_1 &= D_t + \frac{d}{dt} \ln \left( \frac{t^3}{1-t+3t^2} \right), \\
 L_2 &= D_t^2 + 2 \frac{16t^2 + 21t - 4}{t(8t-1)(t+1)} \cdot D_t + 4 \frac{4t^2 + 10t - 3}{(8t-1)(t+1)t^2}.
 \end{aligned}
 \tag{73}$$

Equation (4.38) of [97], when rewritten in a homogeneous way, corresponds to the order-three linear differential operator which is the direct sum  $L'_1 \oplus L'_2$ , of an order-one operator  $L'_1$  and an order-two operator  $L'_2$ :

$$\begin{aligned}
 L'_1 &= D_t + \frac{d}{dt} \ln \left( \frac{(t+1)t^4}{(t-1)(1-2t+4t^2)} \right), \\
 L'_2 &= D_t^2 + 2 \frac{5 - 35t + 16t^2}{(8t-1)(t-1)t} \cdot D_t + 4 \frac{5 - 18t - 41t^2 + 10t^3 + 6t^4 - 4t^5}{(1-8t)(1-t)^2(1+t)^2t^2},
 \end{aligned}
 \tag{74}$$

where these last two operators ( $L_2$  and  $L'_2$ ) of order two are equivalent and actually correspond to a Heun function<sup>54</sup>. A solution of  $L_1 \oplus L_2$  corresponds to a generating function with *integer*

<sup>54</sup> Note that the notation in [97, 99] are Snow's notations which corresponds to a change of sign of the second argument in the Heun functions compared to maple's notations. Also note a sign misprint in [97] ( $t$  in [97])

coefficients, namely:

$$\begin{aligned} & \frac{1}{3t^3} \cdot (\text{Heun}(-1/8, -1/4; -1, -2, 2, -2; -t) - (1 - t + 3t^2)) \\ & = 1 + 2t + 6t^2 + 22t^3 + 92t^4 + 422t^5 + \dots \end{aligned} \tag{75}$$

We have calculated the  $p$ -curvature of the corresponding linear differential operators and found that they are all globally nilpotent.

A large set of other enumerative combinatorics problems for which a linear ODE exists for the generating function, but a representation of that generating function as a multiple integral has *not yet* been found, could be listed [98–102].

### 8. Beyond order-two and three ODEs: staircase polygons and Calabi–Yau type ODEs

Krammer’s counterexample to the first Dwork’s conjecture (reduction of global nilpotence to  ${}_2F_1$  hypergeometric function up to pullback by a rational function) corresponded to a selected Heun function. Let us recall that some  ${}_3F_2$  hypergeometric functions for selected values of the parameters [103] like

$${}_3F_2([\alpha, \beta, \gamma + 1], [\delta, \gamma]; x) \tag{76}$$

have a Heun representation. We want, now, to explore globally nilpotent higher order linear operators beyond the  ${}_2F_1$ , Heun and  ${}_3F_2$  framework.

In appendix D, the linear differential operator for  $\tilde{\chi}_d^{(4)}(t)$  is displayed and factorized in direct sums of an operator of order one, an operator of order three and an operator of order four. The operator of order three is seen to be *equivalent to a symmetric square* of a second-order operator corresponding to a hypergeometric function  ${}_2F_1$ . Similar to the situation described in section 6.2, the order-four linear differential operator is not obviously reducible to a symmetric cube of a  ${}_2F_1$  (even with a pullback), it may however be equivalent to a symmetric cube of a  ${}_2F_1$ . It may well be a  ${}_4F_3$  hypergeometric function with an involved (Hauptmodul, ...) pullback, but, for the moment, we have not been able to decipher this order-four operator. Let us display, in the following, a few more order-four (and more) globally nilpotent operators, related to staircase polygons and Calabi–Yau manifolds.

#### 8.1. Staircase polygons

In an enumerative combinatorics framework other Heun functions can be found for the generating function of the staircase polygons [86, 104].

The Fuchsian differential equations corresponding to the *staircase polygons* generating functions in  $d$  dimensions, that we denote  $\mathcal{Z}_d$  are given in [86]:

$$\begin{aligned} \mathcal{Z}_3 &= D_x^2 + \frac{(1 - 20x + 27x^2)}{x \cdot (1 - x)(1 - 9x)} \cdot D_x - 3 \frac{1 - 3x}{x \cdot (1 - x)(1 - 9x)}, \\ \mathcal{Z}_4 &= D_x^3 + 3 \frac{(1 - 30x + 128x^2)}{x \cdot (1 - 4x)(1 - 16x)} \cdot D_x^2 + \frac{(1 - 68x + 448x^2)}{x^2 \cdot (1 - 4x)(1 - 16x)} \cdot D_x - 4 \frac{1}{x^2 \cdot (1 - 4x)}, \end{aligned}$$

and we give  $\mathcal{Z}_5$  and  $\mathcal{Z}_6$  in appendix B.5.

All these linear operators (up to  $\mathcal{Z}_7$  in [86]) are globally nilpotent: this is a simple consequence of the fact that the generating functions of the staircase polygons are expressed

instead of  $-t$  in [99]). Do note that changing  $-1/4$  into  $+1/4$  in the previous Heun solution, one gets  $\text{Heun}(-1/8, +1/4, -1, -2, 2, -2; -t)/t^3$  which is the solution of a second-order operator almost identical to  $L_2$  in (73), where  $4t^2 + 10t - 3$  is replaced by  $4t^2 + 11t - 3$ . This new operator is *not globally nilpotent*, it differs from  $L_2$  by the so-called [49–52] accessory parameters.

as  $n$ -fold integrals of (very) simple algebraic integrands, and are therefore DFG. We calculated systematically the  $p$ -curvatures of all these Fuchsian linear differential operators (considered modulo the first thousand primes) and found that their characteristic polynomial is  $T^{d-1}$ , where  $d - 1$  is the order of the operator  $\mathcal{Z}_d$ .

The first two staircase linear differential operators  $\mathcal{Z}_3$  and  $\mathcal{Z}_4$  actually correspond to Heun functions. The solution of the Heun linear differential operator  $\mathcal{Z}_3$  and its series expansion has integer coefficients:

$$\begin{aligned} \text{Heun}(1/9, 1/3, 1, 1, 1, 1, x) &= \text{Heun}(9, 3, 1, 1, 1, 1, 9x) \\ &= 1 + 3x + 15x^2 + 93x^3 + 639x^4 + 4653x^5 + 35169x^6 + \dots \end{aligned}$$

The  $\text{Heun}(1/9, 1/3, 1, 1, 1, 1, x)$  solution can also be written in terms of a modular hypergeometric function (corresponding to the weight-1 modular form  $h_6$  in [85]):

$$\begin{aligned} \text{Heun}(1/9, 1/3, 1, 1, 1, 1, x) &= ((3x - 1) \cdot (243x^3 - 243x^2 + 9x - 1))^{-1/4} \cdot {}_2F_1([1/12, 5/12], [1], M_6) \\ \text{with } M_6 &= 1728 \frac{(x - 1)(9x - 1)^3 \cdot x^2}{(3x - 1)^3 \cdot (243x^3 - 243x^2 + 9x - 1)^3}. \end{aligned} \tag{77}$$

Joyce has shown [78] that the square of this Heun function<sup>55</sup> is related (up to quite involved algebraic transformations of the arguments see also equations (23) and (24) in [86]) to the simple-cubic Green function where its DFG nature becomes clear:

$$P(z) = \frac{1}{\pi^3} \cdot \int \int \int_0^\pi \frac{dx_1 dx_2 dx_3}{1 - z/3 \cdot (\cos(x_1) + \cos(x_2) + \cos(x_3))}.$$

This Heun function  $\text{Heun}(1/9, 1/3, 1, 1, 1, 1, x)$ , we now see as a weight-1 modular form, had been seen by Guttman and Prellberg to be equal to the product of two complete elliptic integrals of the first kind  $K$  (up to slightly involved algebraic transformations namely equations<sup>56</sup> (25) and (28) in [86]). As a (curious) byproduct one thus deduces a new relation between a weight-1 modular form and a product of two complete elliptic integrals.

The function  $\text{Heun}(1/4, 1/8, 1/2, 1/2, 1, 1/2, 4x)$  corresponds to the Heun linear differential operator:

$$A_4 = D_x^2 + \left( \frac{1}{x} + \frac{2}{4x - 1} + \frac{8}{16x - 1} \right) \cdot D_x + \frac{2 \cdot (8x - 1)}{(4x - 1)(16x - 1)x}.$$

Its series expansion has *integer* coefficients:

$$\begin{aligned} \text{Heun}(1/4, 1/8, 1/2, 1/2, 1, 1/2, 4x) &= \text{Heun}(4, 1/2, 1/2, 1/2, 1, 1/2, 16x) \\ &= 1 + 2x + 12x^2 + 104x^3 + 1078x^4 + 12\,348x^5 + 150\,528x^6 + \dots \end{aligned}$$

One easily verifies that the *symmetric square* of the previous Heun linear differential operator  $A_4$  is *nothing but*  $\mathcal{Z}_4$ . Actually  $\text{Heun}(1/4, 1/8, 1/2, 1/2, 1, 1/2, 4x)^2$  is a solution of  $\mathcal{Z}_4$ .

To our knowledge this Heun function  $\text{Heun}(1/4, 1/8, 1/2, 1/2, 1, 1/2, 4x)$  does not have an interpretation as a weight-1 modular form (see also [85, 87, 105, 106]). With these two Heun functions we are thus still in a ‘Dworkian’ framework (hypergeometric functions up to a pullback, selected Heun functions, etc).

<sup>55</sup> See also equation (36) in [97] for the generating function of a watermelon counting (union of friendly walkers).

<sup>56</sup> Remarkably, this is also true for  $\text{Heun}(1/4, 1/8, 1/2, 1/2, 1, 1/2, 4t)^2$  (see below) (these are equations (25) and (28) in [86]).

The very nature of  $\mathcal{Z}_5, \mathcal{Z}_6, \mathcal{Z}_7, \dots$  is less clear. In particular, it is far from clear that they can be written in terms of  ${}_2F_1$  hypergeometric functions. Let us introduce some kind of ‘multi-singular’ (beyond the four singularities of a Heun ODE) second-order linear ODE:

$$z_5 = D_x^2 + \left( \frac{1}{x} - \frac{1}{3(1-x)} - \frac{3}{1-9x} - \frac{25}{3(1-25x)} \right) \cdot D_x - \frac{19}{270(1-x)} - \frac{171}{40(1-9x)} - \frac{5875}{216(1-25x)} - \frac{49}{30x} + \frac{1}{90(1-x)^2} + \frac{9}{10(1-9x)^2} + \frac{125}{18(1-25x)^2}. \tag{78}$$

One finds that the explicit expression of  $\mathcal{Z}_5$  is ‘close’ to be a symmetric cube of this ‘multi-singular’ (beyond the four singularities of a Heun ODE) linear differential operator:

$$\mathcal{Z}_5 = \text{Sym}^3(z_5) + \frac{1}{100} \frac{P(x)}{(1-x)^4(1-9x)^4(1-25x)^4 \cdot x^3},$$

where  $P(x)$  is a polynomial of degree 11 with integer numbers. In fact this order-four operator is associated with Calabi–Yau manifolds [107] (see also the following section).

Similarly, introducing the ‘multi-singular’ (beyond the four singularities of a Heun ODE) second-order operator

$$z_6 = D_x^2 + \left( \frac{1}{x} - \frac{1}{1-4x} - \frac{4}{1-16x} - \frac{9}{1-36x} \right) \cdot D_x + \frac{1}{5} \cdot \frac{p(x)}{x \cdot (1-4x)^2(1-16x)^2(1-36x)^2},$$

$$p(x) = -7 + 696x - 22\,224x^2 + 298\,816x^3 - 1603\,584x^4 + 3068\,928x^5, \tag{79}$$

one finds that  $\mathcal{Z}_6$  is (up to some rational functions  $A(x)$  and  $B(x)$ ) ‘close’ to be a symmetric fourth power of this previous ‘multi-singular’ (beyond the four singularities of a Heun ODE) operator (79):

$$\mathcal{Z}_6 = \text{Sym}^4(z_6) + A(x) \cdot D_x + B(x). \tag{80}$$

Let us recall that, even for a second-order operator, it is not so easy, and systematic, to see that a solution reduces to a hypergeometric function  ${}_2F_1$  up to a possibly involved rational pullback and up to a multiplication by some  $N$ th root of a rational function (see (37), (49)). It has to be the case [82] for zero-curvature second-order operators (and this is an important motivation to perform our  $p$ -curvature calculations, to extract, very quickly, the zero-curvature situations like (37)), when we have a basis of algebraic solutions, but this is far from clear, at first sight, for a globally nilpotent operator (see (49)).

When one encounters higher order irreducible linear differential operators like these  $\mathcal{Z}_d$  for  $d \geq 5$  (or other order-three irreducible operators like  $M_3$  in the three-choice polygon problem, see (B.2)), it remains to see<sup>57</sup> if these irreducible operators cannot be symmetric powers of a smaller order operator with an emphasis on second-order operators simply reducing, or not simply reducing, to  ${}_2F_1$  hypergeometric functions.

With these globally nilpotent staircase operators  $\mathcal{Z}_5, \mathcal{Z}_6, \mathcal{Z}_7, \dots$  we are clearly in a win-win situation. In a first scenario, the very simple and regular accumulation of singularities ( $x = 1/n^2$ , for  $n$  even or odd) cannot be wrapped onto three, or four, canonical singularities

<sup>57</sup> To check if an order  $q$  operator is exactly a symmetric power of, say, an order-two operator can be done systematically (pattern matching). To see if an order  $q$  operator is equivalent (homomorphic) to some symmetric power, is less easy and systematic [108–112].

0, 1,  $a$ ,  $\infty$ , and one gets brand new examples of selected<sup>58</sup> ‘multi-singular’ (beyond the four singularities of a Heun ODE) globally nilpotent operators of arbitrary order  $d - 1$ , and fascinating algebraic geometry interpretations (similar to that Krammer gave for the Heun ODE like (5), with Shimura curves) which remain to be found. In a second scenario these staircase generating functions can be expressed as products of complete elliptic integrals (up to involved algebraic transformations) or  ${}_2F_1, {}_3F_2$ , with some (Hauptmodul, . . . ) pullback, and this would provide a set of highly non-trivial effective algebraic geometry results associated with these  $n$ -fold integrals and these enumeration of staircase polygons.

### 8.2. Calabi–Yau type ODEs

Other non-trivial examples of non-trivial globally nilpotent high-order operators can be found with the fourth-order differential equations of the so-called *Calabi–Yau type* [113, 114]. We will not give any detail on the construction of these new types of Calabi–Yau manifolds using conifold transformations from *toric Calabi–Yau hypersurfaces*. We just display some of these fourth-order differential operators.

A first order-four differential operator comes from Kontsevich’s observation [113] that two selected matrices for a quintic and its mirror, actually correspond to monodromy matrices of the Picard–Fuchs operator:

$$\theta^4 - 5^2x \cdot \left(\theta + \frac{1}{5}\right)\left(\theta + \frac{2}{5}\right)\left(\theta + \frac{3}{5}\right)\left(\theta + \frac{4}{5}\right), \tag{81}$$

where

$$\theta = x \cdot \frac{d}{dx}$$

having four solutions which can be expressed in terms of hypergeometric functions  ${}_4F_3$ :

$${}_4F_3\left(\left[\frac{n}{5}, \frac{n}{5}, \frac{n}{5}, \frac{n}{5}\right], \left[\frac{n+i_1}{5}, \frac{n+i_2}{5}, \frac{n+i_3}{5}\right]; \frac{1}{5^2x}\right), \tag{82}$$

where  $n = 1, 2, 3, 4$  and  $i_1, i_2, i_3$  are three integers in the list  $[1, 2, 3, 4]$ . Another example of order-four differential operator which can be found in [113] reads

$$\theta^4 - x \cdot (65\theta^4 + 130\theta^3 + 105\theta^2 + 40\theta + 6) + 4x^2(4\theta + 3)(\theta + 1)^2(4\theta + 5). \tag{83}$$

The critical exponents of (83) are given in [113] in P-Riemann function notations. They are very simple integer or rational numbers  $(3/4, 5/4)$ .

Associated with the diffeomorphisms [114]  $X_{144,120}^{45}$  another order-four differential operator reads

$$\theta^4 - 2x(102\theta^4 + 204\theta^3 + 155\theta^2 + 53\theta + 7) + 4x^2(\theta + 1)^2 \cdot (396\theta^2 + 792\theta + 311) - 784x^3(\theta + 1)(\theta + 2)(2\theta + 1)(2\theta + 5), \tag{84}$$

which, simply written in  $x$ , has a form very similar to the staircase operators  $\mathcal{Z}_5$  given by (B.4) in appendix B.5, namely:

$$D_x^4 + 6 \frac{1568x^2 - 268x + 1}{x(1 - 4x)(1 - 196x)} \cdot D_x^3 + \frac{7 - 2962x + 39260x^2 - 116816x^3}{(1 - 196x)(1 - 4x)^2x^2} \cdot D_x^2 + \frac{1 - 1028x + 22740x^2 - 90944x^3}{(1 - 196x)(1 - 4x)^2x^3} \cdot D_x - 2 \frac{7 + 3920x^2 - 622x}{x^3 \cdot (1 - 196x)(1 - 4x)^2}. \tag{85}$$

<sup>58</sup> Do recall that, for instance, Heun operators are not generically globally nilpotent (see (8)).



Associated with the diffeomorphisms [114]  $X_{200,140}^{51}$  another order-four linear differential operator reads

$$\theta^4 - x(113\theta^4 + 226\theta^3 + 173\theta^2 + 60\theta + 8) - 8x^2(\theta + 1)^2 \cdot (119\theta^2 + 238\theta + 92) - 484x^3(\theta + 1)(\theta + 2)(2\theta + 1)(2\theta + 5),$$

which, simply written in  $x$ , has, again, a form very similar to the staircase operator  $\mathcal{Z}_5$  given by (B.4) in appendix B.5, or to the previous order-four operator (85), but with the denominators  $(1 - 196x)(1 - 4x)^r$  of (85) changed into  $(1 - 121x)(1 + 4x)^r$ .

We have calculated the  $p$ -curvature of these four order-four linear differential operators (81), (83), (85) and (86) (modulo the first thousand primes) and found that the characteristic polynomial, as well as the minimal polynomial, read  $T^4$ , and that the Jordan-block reduction of the  $4 \times 4$   $p$ -curvature matrix reads

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{86}$$

### 9. Beyond holonomic functions: ratio of holonomic functions

In [115, 116] it has been shown that the enumeration of three-dimensional convex polygons can be written as *ratio of holonomic functions*.

For instance the equation in proposition 4.12 in [116], or the equation just before the conclusion in [115], gave the perimeter generating function for three-dimensional oriented convex polygons as the ratio

$$\frac{N_1}{S_3(x)}, \quad \text{with } N_1 = A_1(u) \cdot S_3(x) + A_2(u), \tag{87}$$

where  $u$  denotes the square root of  $x$ , where  $A_1(x)$  and  $A_2(x)$  denote algebraic expressions of  $u$  and where  $S_3(x)$  denotes the solution of the previous staircase operator  $\mathcal{Z}_3$  given by (77). Such a numerator  $N_1$ , is also of the type considered in section 7, namely functions for which an integral representation has not yet been found. Do note that the *ratio* of holonomic functions is far from being holonomic<sup>59</sup>, as can be seen on the solutions of the Chazy III nonlinear ODE (see, for instance, (3.37) in [117], and p 1878 in [4]). In terms of  $x$  and not its square root  $u$ , the algebraic expression  $A_2$  is the solution of a linear differential operator of order four,  $L_4$  (direct sum of an operator of order three and an operator of order one). The algebraic expression  $A_1$  is the solution of a linear differential operator of order five,  $L_5$  (direct sum of an operator of order two and three operators of order one), the product  $A_1(u) \cdot S_3(x)$  being the solution of a linear differential operator of order ten  $L_{10}$  (symmetric product of  $\mathcal{Z}_3$  and  $L_5$ ). The numerator  $N_1$  is the solution of an order-14 linear operator that can be obtained as the LCLM of  $L_{10}$  and  $L_4$ . We have calculated<sup>60</sup> the  $p$ -curvature for the *first two hundred primes*, of this order-14 operator (the degree of the polynomial coefficients is 320) and found that its characteristic polynomial is  $T^{14}$ , its minimal polynomial being  $T^2$ . The global nilpotence of this order-14 operator is a straight consequence of the expression of  $N_1$  in (87). Examples like the enumeration of three-dimensional convex polygon suggest to seek for new classes of solutions that are, not only the ratio of holonomic functions, but *the ratio of solutions of*

<sup>59</sup> In contrast, the product of two holonomic functions is holonomic.

<sup>60</sup> Note that these  $p$ -curvature calculations are very quickly performed, when the factorization (or LCLM factorization) of such order-14 operators is easily reaching the limits of our 32-Gigas computer facilities.

two globally nilpotent operators ‘algebraically equivalent’ in the sense of relation (87), such ratio of periods probably having a (modular) interpretation as  $\tau$  functions and Painlevé-like Picard–Fuchs deformations [118].

### 10. Beyond global nilpotence: linear differential operators with irregular singularities

We have encountered in previous publications [74]  $n$ -fold integrals annihilated by (minimal) linear differential operators which are obviously globally nilpotent, namely the two-point correlation functions and the form factors of the off-critical lattice Ising model. The linear differential operators of the form factors have a nice ‘Russian-doll’ structure (see (18)). The linear differential operators  $F_j(N)$  occurring [74] in these factorized Russian-doll form were seen to be equivalent to symmetric powers of the second-order linear differential  $L_E$  corresponding to the complete elliptic integral of the second kind  $E$ . Consequently, they are obviously globally nilpotent. The *scaling limit* of these linear differential operators also exhibit a ‘Russian-doll’ structure, but they are not globally nilpotent.

The scaling limit of the  $f_{N,N}^{(n)}$ ’s amounts, on the functions, and on the corresponding differential operators, to taking the limit  $N \rightarrow \infty$  and  $t \rightarrow 1$ , keeping the limit  $x = N \cdot (1 - t)$  finite, or in other words, to performing the change of variables  $t = 1 - x/N$ , keeping only the leading term in  $N$ . Performing these straightforward calculations, the linear differential operators in  $t$  for the  $f_{N,N}^{(n)}$ ’s where  $N$  was a parameter, become linear differential operators in the scaling variable  $x$ .

Calling  $F_j^{\text{scal}}$  the scaling limit of the operator  $F_j(N)$  we found [74] for  $j$  odd, that

$$\begin{aligned} F_1^{\text{scal}} &= L_2^{\text{scal}}, \\ F_3^{\text{scal}} &= L_4^{\text{scal}} \cdot L_2^{\text{scal}}, \\ F_5^{\text{scal}} &= L_6^{\text{scal}} \cdot L_4^{\text{scal}} \cdot L_2^{\text{scal}}, \quad \dots \end{aligned}$$

where

$$\begin{aligned} L_4^{\text{scal}} &= 16x^4 D_x^4 + 96x^3 D_x^3 + 40(2 - x^2)x^2 D_x^2 + 8(x^2 - 2)x D_x + 9x^4 - 8x^2 + 16, \\ L_2^{\text{scal}} &= 4x \cdot D_x^2 + 4D_x - x, \end{aligned} \tag{88}$$

and  $L_{10}^{\text{scal}}, L_8^{\text{scal}}, L_6^{\text{scal}}$  are given in [74]. Similar relations occur for  $j$  even [74]. Thus, we see that the scaled operators  $F_j^{\text{scal}}$  have a ‘Russian-doll’ structure inherited from the lattice operators  $F_j(N)$ .

Consider the linear differential operator corresponding to the modified Bessel function  $\text{Bessel}(n, x/2)$  for  $n = 0$ , namely:

$$B = D_x^2 + \frac{D_x}{x} - \frac{1}{4}. \tag{89}$$

We recognize, in this linear differential operator, the exact identification with the scaled differential operator  $F_1^{\text{scal}} = L_2^{\text{scal}}$ . We find that the symmetric square of the linear differential operator  $B$ , and the scaled operator  $L_3^{\text{scal}}$  are equivalent, the symmetric third power of the linear differential operator  $B$ , and the scaled operator  $L_4^{\text{scal}}$  are equivalent, and, more generally, the symmetric  $j$ th power of (89) and the scaled operator  $L_{j+1}^{\text{scal}}$  are equivalent,  $L_{j+1}^{\text{scal}} \simeq \text{Sym}^j(B)$ .

Global nilpotence implies Fuchsianity. The scaling limit generates a *confluence of the regular singular points* [119, 120] we had on the lattice, yielding linear differential operators, which are *not* Fuchsian anymore because of an *irregular singular point at infinity*: we are leaving the universe of  $G$ -functions for the universe of ‘Hamburger’ functions [21].

Let us explore, however, the  $p$ -curvatures of the previous *non-Fuchsian* linear differential operators which correspond to scaling limits of globally nilpotent linear differential operators. The calculations give, modulo the prime  $p$ , the following characteristic polynomial for  $L_2^{\text{scal}}$ :

$$\left(T + \frac{p-1}{2}\right) \cdot \left(T + \frac{p+1}{2}\right) \tag{90}$$

and the following characteristic polynomial for  $L_4^{\text{scal}}$ :

$$\left(T + \frac{p-3}{2}\right) \cdot \left(T + \frac{p-1}{2}\right) \cdot \left(T + \frac{p+1}{2}\right) \cdot \left(T + \frac{p+3}{2}\right). \tag{91}$$

We have also calculated the  $p$ -curvatures of  $L_3^{\text{scal}}$  and the corresponding characteristic polynomials. For almost all primes, this characteristic polynomial has a very simple expression one can see as a *simple deformation* of the characteristic polynomial of a nilpotent operator:  $T^3 - T = (T - 1) \cdot T \cdot (T + 1)$ . Similar calculations performed for  $L_5^{\text{scal}}$ ,  $L_6^{\text{scal}}$ ,  $L_7^{\text{scal}}$ , up to  $L_{10}^{\text{scal}}$  give the following results for the characteristic polynomial of the corresponding  $p$ -curvature:

$$\left(T - \frac{n-1}{2}\right) \cdots (T-2) \cdots (T-1) \cdot T \cdot (T+1) \cdots (T+2) \cdots \left(T + \frac{n-1}{2}\right) \tag{92}$$

for  $L_n^{\text{scal}}$  with  $n$  odd and

$$\prod_{i=1-n/2}^{i=n/2} \left(T + \frac{p-1}{2} + i\right), \tag{93}$$

for  $L_n^{\text{scal}}$  with  $n$  even. All these calculations have been performed for all the primes  $p < 100$ .

A remarkable structure ‘beyond global nilpotence’ clearly remains to be discovered by mathematicians for the functions of ‘Hamburger<sup>61</sup> type’ (one irregular singular point at infinity) that occur in field theory, or, more simply, in the scaling limit of DFG holonomic functions of lattice problems.

### 11. Conclusion

One can probably conjecture that when the generating functions of the various problems of enumerative combinatorics are found to be solutions of Fuchsian ODEs, quite systematically the corresponding linear differential operators are globally nilpotent, these holonomic functions being ‘DFG’, their rewriting in terms of  $n$ -fold integrals being just a question of time, work and/or stamina. The generating function of the perimeter three-choice polygon, of the directed compact percolation or of the vicious walkers are such examples. In this paper, we have studied a quite large number of  $n$ -fold integrals of algebraic integrands and their corresponding Fuchsian ODEs. In particular, we looked at the  $p$ -curvatures of their factors, not to see if these linear differential factors were globally nilpotent<sup>62</sup>, but to understand *how* these differential factors ‘succeed’ to be globally nilpotent. One must keep in mind that the Fuchsian ODEs for the  $n$ -fold integrals of theoretical physics (Feynman diagrams [62], ...) are generically of quite large orders (as an example the minimal order ODE for  $\chi^{(5)}$  is of order 33, see [5]). Since the corresponding minimal order ODEs are necessarily globally nilpotent because they are DFG, the question one can ask is *how* an order 23, 33, 50

<sup>61</sup> To sum up quite brutally the situation, one may say [19, 20] that almost all the special functions occurring in theoretical physics are either  $G$ -functions for Fuchsian ODEs or ‘Hamburger’ functions [21] when an irregular singularity occurs.

<sup>62</sup> This question has been solved by mathematicians: these  $n$ -fold integrals are holonomic functions with rational critical exponents, and are even DFG.

linear differential operator succeeds to be globally nilpotent? Throughout all the examples displayed in this paper we have seen that the (minimal order) linear differential operators of quite large order actually factorize into products (sometimes direct sums and products) of linear differential operators of smaller orders (one, two, sometimes three and four). The global nilpotence of the order-one operators just corresponds to Wronskians that are  $N$ th roots of rational functions, most of the order-two linear differential operators and, for instance, the order-three and four operators occurring in the factorizations of  $\chi^{(3)}$  and  $\chi^{(4)}$ , having a typically ‘Dworkian’ interpretation since they correspond to either the second-order linear differential operator associated with the complete elliptic integrals (of the first or second kind), or equivalently to  ${}_2F_1$  hypergeometric function or to symmetric powers (square and cube) of these second-order linear differential operators. More remarkably, other second-order linear differential operators were found to correspond not only to globally nilpotent operators but to zero curvature operators. The solutions are algebraic functions corresponding to selected algebraic curves. For instance, we encountered *genus 6 and genus 5* algebraic curves (for  $\Phi_D^{(5)}$  and  $\Phi_D^{(6)}$  respectively), their roots being expressed as complete elliptic integrals of the third kind with a ‘characteristic’ corresponding to *genus 3* curves. As far as algebraic curves are concerned, we also gave many examples of zero curvature linear differential operators of different orders corresponding exactly to algebraic functions associated to *modular curves*, namely the  $\lambda$ -extensions  $C(N, N; \lambda)$  for some selected values [74] of  $\lambda$ . From a ‘Dworkian viewpoint’ let us recall that these algebraic functions can be expressed as pullbacks of  ${}_2F_1$  hypergeometric functions. We also gave many examples of zero curvature linear differential operators of different orders corresponding exactly to algebraic functions associated with *modular curves*. As far as second-order linear differential operators are concerned, the most spectacular example came from the linear differential operator  $Z_2$  occurring in the factorization of the linear differential operator for  $\chi^{(3)}$ . The global nilpotence of that operator  $Z_2$  was seen to correspond to a highly non-trivial pullback of  ${}_2F_1$  hypergeometric function, namely the *weight-1 modular forms*  $h_6$ ! Most of these examples correspond to  $n$ -fold integrals associated with the Ising models or more generally  $n$ -fold integrals of the so-called Ising class.

Among all the globally nilpotent operators we displayed in this paper, other  $n$ -fold examples came from enumerative combinatorics and others from Picard–Fuchs (Gauss–Manin connection [7]) constructions on (mirror) Calabi–Yau hypersurfaces with conifold singularities. In the first case (enumerative combinatorics) only two examples were clearly ‘Dworkian’ the  $n$ -fold integral being expressed alternatively as a Heun function or as a pullback of a  ${}_2F_1$  hypergeometric function. All the other examples seem to go beyond a strict hypergeometric ‘Dworkian’ framework. One seems to explore, similar to Krammer counterexamples, Heun functions that cannot be reduced to  ${}_2F_1$  hypergeometric functions, and, more generally, holonomic functions corresponding to linear differential operators of order two, three, four,  $\dots$ , with, at first sight, *many more singularities* than 0, 1 and  $\infty$ .

The details and the richness of the situations, and structures, encountered with our physical examples were certainly not obviously expected from the DFG diagnostic.

Understanding *how* linear differential operators are globally nilpotent led us to discover different structures on various algebraic varieties (elliptic curves and complete elliptic integrals, algebraic curves that are *modular curves*, *weight-1 modular forms*,  $\dots$ ) that provide a deeper understanding of the underlying mathematical structures ‘hidden’ in the physics problems we study. When we see that, to sum up things brutally, the global nilpotence of the linear differential operator for  $\chi^{(3)}$  is inherited from the global nilpotence of  $Z_2$  which corresponds to the weight-1 modular forms  $h_6$ , one understands the ‘complexity’ of the holonomic function  $\chi^{(3)}$ , totally, and utterly, differently. An interesting generalization of the previous weight-1 modular form was found with  $\tilde{\chi}_d^{(3)}$  and equation (64), with the occurrence of a  ${}_3F_2$

hypergeometric function with a *Hauptmodul pullback*. This reinforces the viewpoint we tried to promote that a (serious) theory of the Ising model embed all the theory of elliptic curves (modular curves, modular forms, ...). The last Calabi–Yau examples confirm that viewpoint: discovering the underlying algebraic varieties (or projective spaces minus some singular sets [121, 122]), is fundamental. What are these algebraic varieties curves, surfaces and higher dimensional algebraic varieties? Going a bit further beyond the simple mantra ‘it is derived from geometry’, we tried, and often succeeded, to find *explicitly* the structures of effective algebraic geometry that are the ‘deus ex machina’ of our theoretical physics problems.

One can now propose the following systematic, and quite algorithmic, study of every  $n$ -fold integral of algebraic expression encountered in theoretical physics: first generate large series expansions of these  $n$ -fold integrals to find out the linear differential operators that annihilate these series, then get the minimal order differential operators, then factorize and LCLM-factorize, as much as possible, these minimal order differential operators, then examine the irreducible factors to see if they are not equivalent to symmetric powers of smaller order operators, then calculate the corresponding  $p$ -curvatures<sup>63</sup> of all these smaller order irreducible operators to see if they have zero curvature, or if they are globally nilpotent, and, finally, examine all these smaller order irreducible operators to find out if they correspond to  ${}_2F_1, {}_3F_2, {}_4F_3, \dots$  hypergeometric functions up to a rational pullback.

Saying that theoretical physics should eventually reduce to classification of singular varieties [121, 122], thus reducing most of it to effective algebraic geometry, is certainly a too drastic simplification. With section 10 we see that some nice generalization of the notion of global nilpotence does exist, and needs to be explored, for the Hamburger functions [21] that typically occur in physics (linear ODEs with one irregular singularity, for instance at  $\infty$ ). Let us just say that an effective algebraic geometry viewpoint of lattice statistical mechanics, enumerative combinatorics, particle physics, solid state physics, theoretical physics, hopefully yielding the emergence of a new ‘Algebraic Statistical Mechanics’ is certainly a step in the good direction.

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## Appendix A. Factorizations of multiple integrals linked to $\zeta(3)$

In Apéry’s proof of the irrationality of  $\zeta(3)$  a crucial role is played by the linear differential operator [123]:

$$(t^2 - 34t + 1) \cdot t^2 \cdot D_t^3 + (6t^2 - 153t + 3) \cdot t \cdot D_t^2 + (7t^2 - 112t + 1) \cdot D_t + (t - 5), \quad (\text{A.1})$$

<sup>63</sup> For operators of quite large order and degree of the polynomial coefficients, the  $p$ -curvature calculations are much quicker than the factorization and LCLM factorization.

this operator being linked to the modularity of the algebraic variety:

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0.$$

Operator (A.1) is, in fact, the symmetric square of the second-order operator [124]:

$$4 \cdot (t^2 - 34t + 1) \cdot t \cdot D_t^2 + 4 \cdot (2t^2 - 51t + 1) \cdot D_t + (t - 10). \tag{A.2}$$

If one introduces the change of variable

$$t = \frac{x \cdot (1 - 9x)}{1 - x}, \tag{A.3}$$

the second-order operator (A.2) becomes

$$D_x^2 + \frac{1 - 18x}{(1 - 9x)x} \cdot D_x - \frac{1}{4} \cdot \frac{10 - 11x + 9x^2}{(1 - 9x)(1 - x)^2x}. \tag{A.4}$$

Considering the solution up to a multiplicative square root  $(1 - x)^{1/2}$ , amounts to performing the symmetric product of this operator (A.4) with the order-one operator  $D_x + 1/2/(x - 1)$  and transforms (A.4) into the second-order operator  $L_x$ :

$$D_x^2 + \frac{1 - 20x + 27x^2}{(1 - 9x)(1 - x)x} \cdot D_x - 3 \cdot \frac{1 - 3x}{(1 - 9x)(1 - x)x},$$

which is nothing but  $\mathcal{Z}_3$  (see section 8.1, or equivalent to the operators  $Z_2$  and  $M_2$  of  $\Phi_H^{(3)}$ ) that occurred in the staircase polygons, and has the Heun function solution  $(1 - x)^{1/2} \cdot \text{Heun}(1/9, 1/3; 1, 1, 1, 1; x)$  that already occurred in  $Z_2$  and  $M_2$ . Again we have here [124] the occurrence of a Picard–Fuchs equation<sup>64</sup> for the modular family of elliptic curves associated with  $\Gamma_1(6)$ , that is the weight-1 modular form  $h_6$ .

In [59] we also obtained an order-four Fuchsian linear differential equation (also related to the analysis of  $\zeta(3)$ ) which factorizes<sup>65</sup> in four order-one differential operators ( $Dx$  denotes  $d/dx$ ):

$$\begin{aligned} L_n &= Dx^4 + \frac{2(3x - 1)}{(x - 1)x} \cdot Dx^3 + \frac{(7x^2 + (n^2 + n - 5)x - 2n(n + 1))}{(x - 1)^2x^2} \cdot Dx^2 \\ &\quad + \frac{(x^2 + 2n(n + 1))}{(x - 1)^2x^3} \cdot Dx \\ &\quad + \frac{n \cdot (n + 1) \cdot ((n^2 + n + 1)x + (n - 1)(n + 2))}{(x - 1)^2x^4} \\ &= \left( Dx + \frac{d \ln(A_1)}{dx} \right) \cdot \left( Dx + \frac{d \ln(A_2)}{dx} \right) \\ &\quad \times \left( Dx + \frac{d \ln(A_3)}{dx} \right) \cdot \left( Dx + \frac{d \ln(A_4)}{dx} \right). \end{aligned} \tag{A.5}$$

These order-one linear differential operators *have rational solutions*. Such factorization into order-one linear differential operators with rational solutions is a characteristic of factorization of globally nilpotent operators when they are considered modulo a prime. Here, remarkably,

<sup>64</sup> Similarly [125], one also has Gauss–Manin systems of Shimura families of Abelian surfaces having multiplication by a quaternion algebra over  $\mathbb{Q}$ .

<sup>65</sup> Note a misprint in [59] one should read  $\ln A_i$ , instead of  $A_i$ , in the equations defining the  $A_i$  after equation (H.2) in [60, 70].

such a factorization takes place for the exact operators over  $\mathbb{Q}$  and *not only mod prime* ! The Jordan-block reduction of the  $p$ -curvature of the operator (A.1) read respectively:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A.6}$$

when it gives matrix (86) for (A.5) for any integer  $n$ . For any integer  $n$  the characteristic polynomial, as well as the minimal polynomial of the  $p$  curvature of (A.5) reads  $T^4$ . For  $n$  a non-integer parameter the characteristic polynomial reads

$$\begin{aligned} & T^4 + (x - 2)^p \cdot U \cdot T^2 + ((x + 1)(x - 1)^2)^p \cdot U^2, \\ & \text{with } U = A_n^2 \cdot \left( \frac{(x - 1)^2 x^6}{(n - 1)^4 n^8} \right)^p, \quad \text{where} \\ & A_n = (n - 1) \cdot n \cdot (n + 1) \cdot (n + 2) \cdots (n + p - 2), \end{aligned} \tag{A.7}$$

where  $A_n$ , as it should, vanishes modulo the prime  $p$ , when  $n$  is an integer.

### Appendix B. Display of miscellaneous Fuchsian linear operators of the paper

#### B.1. Operators $M_2$ (resp. $M_3$ ) for three-choice polygons

The order-two (resp. three) operator  $M_2$  (resp.  $M_3$ ) occurring in the factorization (70) of the order-six linear differential operator  $M_6$  associated with the three-choice polygons generating function (see section 7.1) reads

$$M_2 = D_x^2 + \frac{P_{11}}{(x - 1) \cdot P_{12} \cdot x} \cdot D_x + \frac{P_{21}}{4 \cdot P_{22}}, \tag{B.1}$$

with

$$\begin{aligned} P_{11} &= 3437x^7 - 1341x^6 + 4188x^5 - 24\,160x^4 + 38\,400x^3 \\ &\quad + 10\,752x^2 - 34\,816x + 12\,288, \\ P_{12} &= 3437x^6 - 5826x^5 + 5280x^4 - 7360x^3 + 7680x^2 + 3072x - 4096, \\ P_{21} &= 24\,059x^9 + 35\,756x^8 + 116\,792x^7 - 480\,784x^6 + 693\,824x^5 \\ &\quad - 2361\,856x^4 + 2886\,656x^3 - 3739\,648x^2 + 3670\,016x - 1376\,256, \\ P_{22} &= -(16 + 4x + 7x^2) \cdot (1 - x)^2 \cdot x \cdot P_{12}, \end{aligned}$$

and

$$M_3 = P(x) \cdot D_x^3 + \dots, \tag{B.2}$$

where  $P(x)$  denotes the (head) polynomial

$$\begin{aligned} P(x) &= 4x^3(1 + x)(1 - x)^2(4 + x^2)(\times(16 + 4x + 7x^2) \cdot P_{12}(x)^3 \cdot Q(x), \\ Q(x) &= 116\,620x^{12} - 39\,739x^{11} + 2816\,770x^{10} - 4827\,228x^9 \\ &\quad - 5350\,720x^8 + 12\,343\,408x^7 + 473\,056x^6 - 13\,436\,096x^5 \\ &\quad + 9007\,872x^4 + 1064\,960x^3 - 1421\,312x^2 - 327\,680x - 65\,536. \end{aligned}$$

#### B.2. The order-two operator $M_2$ in $L_{\Phi_H^{(3)}}$

The order-two operator  $M_2$  occurring in the factorization  $L_{\Phi_H^{(3)}} = M_3 \cdot M_2$  reads

$$p_2 \cdot M_2 = p_2 \cdot Dx^2 + (x - 1) \cdot p_1 \cdot Dx + 3 \cdot p_0,$$



with

$$\begin{aligned} p_2 &= (x - 4)(x - 2)(x - 1)^2 \cdot (2 + x)(4 + 3x + x^2) \cdot x, \\ p_1 &= -64 + 128x + 196x^2 + 20x^3 - 57x^4 - 14x^5 + 7x^6, \\ p_0 &= 33x^3 - 16 - 20x + 44x^2 - 11x^4 - 9x^5 + 3x^6, \end{aligned}$$

and

$$\begin{aligned} M_3 &= q_3 \cdot D^3 + \dots, \\ q_3 &= (x - 2)^2(x - 1) \cdot x^2 \cdot (1 + x)^2 \cdot p_2 \cdot r_3, \quad \text{where} \\ r_3 &= 1280 - 1344x - 6848x^2 - 21456x^3 + 82416x^4 + 74876x^5 \\ &\quad - 44684x^6 - 48873x^7 + 32112x^8 + 25252x^9 + 1728x^{10} \\ &\quad - 1918x^{11} + 648x^{12} + 120x^{13} + 4x^{14} - x^{15}. \end{aligned}$$

Note that  $x$  here is actually  $4w$ . The set of singularities have a  $w \leftrightarrow 1/4/w$  covariance, that is a  $x \leftrightarrow 4/x$  covariance:

$$p_2\left(\frac{4}{x}\right) = 256 \cdot \frac{4 - x}{(1 - x) \cdot x^9} \cdot p_2(x)$$

*B.3. The order-four operator  $M_4$  in  $L_{\Phi_H^{(4)}}$*

The order-four operator  $M_4$  occurring in the factorization  $L_{\Phi_H^{(4)}} = M_4 \cdot K_1 \cdot Z_1$  reads

$$\begin{aligned} M_4 &= q_4 \cdot D_x^4 + q_3 \cdot D_x^3 + q_2 \cdot D_x^2 + q_1 \cdot D_x + q_0, \quad \text{with} \\ q_4 &= 16(4 - x)(1 - x)^4 \cdot Q_4, \\ Q_4 &= -128 - 2233x + 2847x^2 - 3143x^3 + 3601x^4 - 144x^5 + 64x^6, \\ q_3 &= 32x^3(1 - x)^3 \cdot Q_3, \\ Q_3 &= 768x^8 - 4712x^7 + 54621x^6 - 226585x^5 + 271255x^4 \\ &\quad - 253247x^3 + 190228x^2 - 45848x - 3328, \\ q_2 &= -8x^2(1 - x)^2 \cdot Q_2, \\ Q_2 &= 23360x^9 - 140752x^8 + 1814065x^7 - 7479930x^6 + 10944040x^5 \\ &\quad - 11262286x^4 + 9445431x^3 - 4048776x^2 + 419280x + 47104, \\ q_1 &= 8x(1 - x) \cdot Q_1, \\ Q_1 &= 64640x^{10} - 382600x^9 + 5520835x^8 - 22754401x^7 + 38212402x^6 - 43444138x^5 \\ &\quad + 39867319x^4 - 22329197x^3 + 5075028x^2 - 21248x - 47104, \\ q_0 &= -65536 + 1444096x + 4876704x^2 - 79483588x^3 + 250389985x^4 \\ &\quad + 382946518x^6 - 307163242x^7 - 380545497x^5 + 165955737x^8 \\ &\quad - 40044089x^9 + 2440592x^{10} - 419904x^{11}. \end{aligned}$$

*B.4. The second-order operator  $N_1$  in  $L_{\Phi_D^{(5)}}$*

The second-order operator  $N_1$  occurring in the factorization of  $L_{\Phi_D^{(5)}}$  reads

$$\begin{aligned} N_1 &= P_2 \cdot x \cdot D_x^2 + P_1 \cdot D_x + x^2 \cdot P_0, \\ P_2 &= -2(1 + 8x + 20x^2 + 15x^3 + 4x^4)(1 - x - 3x^2 + 4x^3) \\ &\quad \times (1 + 2x - 4x^2)(1 - 3x + x^2)(1 + 2x)(x - 1)(1 + x) \cdot p_2 \end{aligned}$$

$$\begin{aligned}
 p_2 &= 608x^{16} + 88x^{15} - 3092x^{14} - 7329x^{13} + 2156x^{12} + 15\,088x^{11} \\
 &\quad + 7054x^{10} - 3476x^9 - 15\,856x^8 - 6198x^7 + 7668x^6 + 4064x^5 \\
 &\quad - 1014x^4 - 732x^3 + 36x^2 + 63x + 8, \\
 P_1 &= -16 - 188x - 592x^2 - 384x^3 + 48\,895\,500x^{11} - 24\,850\,920x^{12} \\
 &\quad - 216\,496\,824x^{13} + 526\,808x^7 - 24\,630\,720x^{14} - 203\,254\,764x^{23} \\
 &\quad + 526\,770\,352x^{15} + 243\,268\,011x^{16} + 34\,759\,480x^{24} + 91\,207\,968x^{25} \\
 &\quad + 16\,933\,696x^{26} - 13\,056\,000x^{27} - 5926\,400x^{28} + 112\,640x^{29} \\
 &\quad + 622\,592x^{30} + 80\,791\,756x^{21} - 255\,450\,647x^{22} - 7415x^4 \\
 &\quad - 1427\,696x^8 - 6112\,060x^9 + 10\,047\,555x^{10} - 686\,378\,692x^{17} \\
 &\quad + 511\,457\,760x^{20} - 507\,535\,276x^{18} + 380\,400\,540x^{19} - 33\,404x^5 \\
 &\quad + 113\,836x^6, \\
 P_0 &= -144 - 1344x - 4416\,004x^{23} - 7073\,308x^{11} + 223\,379\,312x^{12} \\
 &\quad + 107\,333\,356x^{13} + 66\,480x^3 + 1927\,564x^7 + 2688x^2 \\
 &\quad - 364\,322\,176x^{14} - 265\,131\,272x^{15} + 279\,826\,283x^{16} \\
 &\quad - 10\,176\,392x^{24} - 2082\,592x^{25} + 67\,584x^{26} + 311\,296x^{27} \\
 &\quad + 39\,019\,332x^{21} + 45\,440\,155x^{22} + 176\,313x^4 + 14\,440\,640x^8 \\
 &\quad - 6309\,980x^9 - 73\,169\,703x^{10} + 282\,707\,420x^{17} - 61\,145\,056x^{20} \\
 &\quad - 44\,575\,908x^{18} - 149\,321\,200x^{19} - 446\,364x^5 - 2110\,212x^6. \tag{B.3}
 \end{aligned}$$

*B.5. The order four and five operators for the staircase polygons*

Let us give the order-four and five operators  $\mathcal{Z}_5$  and for  $\mathcal{Z}_6$  the staircase polygons generating functions [86]:

$$\begin{aligned}
 \mathcal{Z}_5 &= D_x^4 + 2 \frac{(3 - 140x + 1295x^2 - 1350x^3)}{x(1-x)(1-9x)(1-25x)} \cdot D_x^3 + \frac{(7 - 518x + 6501x^2 - 8550x^3)}{x^2 \cdot (1-x)(1-9x)(1-25x)} \cdot D_x^2 \\
 &\quad + \frac{(1 - 196x + 3963x^2 - 7200x^3)}{x^3(1-x)(1-9x)(1-25x)} \cdot D_x - 5 \frac{1 - 57x + 180x^2}{x^3(1-x)(1-9x)(1-25x)}, \\
 \mathcal{Z}_6 &= D_x^5 + 10 \frac{1 - 70x + 1176x^2 - 4032x^3}{x(1-4x)(1-16x)(1-36x)} \cdot D_x^4 + \frac{25 - 2408x + 51\,196x^2 - 211\,968x^3}{x^2(1-4x)(1-16x)(1-36x)} \cdot D_x^3 \\
 &\quad + 3 \frac{5 - 812x + 23\,992x^2 - 126\,720x^3}{x^3(1-4x)(1-16x)(1-36x)} \cdot D_x^2 + \frac{1 - 516x + 25\,956x^2 - 193\,536x^3}{x^4(1-4x)(1-16x)(1-36x)} \cdot D_x \\
 &\quad - 6 \frac{1 - 170x + 2304x^2}{x^4(1-4x)(1-16x)(1-36x)}. \tag{B.4}
 \end{aligned}$$

**Appendix C. Exponents of Fuchsian linear ODEs are generically algebraic numbers, not rational numbers**

*C.1. ‘Lattice’ Fuchsian ODEs*

Keeping in mind some mainstream [126] conformal theory prejudice, the fact that critical exponents (for the ferromagnetic/antiferromagnetic critical points) are rational numbers is too often taken for granted. However, here, we have a *much stronger result*: in all our previous

calculations, the critical exponents *all* appear to be *rational numbers for all the singularities of these ODEs* (in our  $n$ -fold integrals of the Ising class, the exponents are even half-integers). Considering the polynomial coefficients in front of the successive derivatives in all our Fuchsian linear ODEs, one could wrongly imagine that this rational number exponent result is a straight consequence of the fact that these polynomial have, themselves, *integer coefficients*, this property being straightforwardly inherited from the enumerative combinatorics nature of the lattice problem. In that respect the simplest example of a Fuchsian linear differential operator certainly corresponds to the Gauss hypergeometric second-order differential operator ( $D_x$  denotes the derivative  $d/dx$ )

$$x \cdot (1 - x) \cdot D_x^2 + (c - (a + b + 1) \cdot x) \cdot D_x - a b, \tag{C.1}$$

which has the following indicial polynomials for the regular singular points  $x = 0, 1, \infty$ :

$$r \cdot (r - 1 + c), \quad r \cdot (r + ab/c), \quad r \cdot (r + 1 - c). \tag{C.2}$$

On these indicial polynomials one sees clearly that exponents being rational numbers is straightforwardly inherited from the rational character of the coefficients of (C.1). More generally for Fuchsian linear ODEs of arbitrary order it can easily be shown, for a singular point which is a rational number<sup>66</sup> and *occurring in the ODE with multiplicity one*, that its exponents are necessarily rational numbers.

However, it is important to underline that Fuchsian linear ODEs with integer coefficients *do not have necessarily rational number exponents*. Generically exponents of such integer coefficients Fuchsian linear ODE are *algebraic numbers* (algebraic over  $\mathbb{Q}$ ) *not rational numbers*.

C.2. ‘Lattice’ Fuchsian ODEs with algebraic numbers but not rational exponents

Let us consider, for instance, the order-four linear differential operator:

$$(x - 1)^2 x^2 \cdot D_x^4 + 4 \cdot (1 - 2x)(1 - x) \cdot x \cdot D_x^3 + (1 - 6x + 6x^2) \cdot D_x^2 + 6 \cdot (1 - 2x) \cdot D_x + 12. \tag{C.3}$$

This linear differential operator is Fuchsian. It has three regular singular points  $x = 0, x = 1$  and  $x = \infty$ , some of its exponents being algebraic numbers simply expressed in terms of the golden number as can be seen on the indicial polynomial corresponding respectively to  $x = 0, 1, \infty$ :

$$r \cdot (r - 1) \cdot (r^2 - r - 1), \quad r \cdot (r - 1) \cdot (r^2 - r - 1), \\ (r - 3) (r - 2) (r + 1) (r + 2),$$

as well as on the solutions of this linear operator:

$$(x - 1)^{-1/2 \cdot \sqrt{5} + 1/2} \cdot x^{1/2 + 1/2 \cdot \sqrt{5}} \cdot (-\sqrt{5} - 3 + 6x), \\ x^{-1/2 \cdot \sqrt{5} + 1/2} (x - 1)^{1/2 + 1/2 \cdot \sqrt{5}} \cdot (-3 + 6x + \sqrt{5}), \tag{C.4} \\ (3x - 1) \cdot x, \quad 1 - 6x^2.$$

It is interesting to calculate the  $p$ -curvature of this Fuchsian operator with a rational Wronskian  $1/((x - 1)^4 \cdot x^4)$ , but *with non-rational critical exponents* (which cannot, therefore, be globally nilpotent). For an infinite number of primes the  $p$ -curvature is not nilpotent (*a fortiori* zero). However, for a subset of prime numbers (for instance,  $p = 11, 19, 29, 31, 41, 59, \dots, 281, 311, \dots$ ) one finds a zero  $p$ -curvature. A heuristic

<sup>66</sup> And we have many examples [5], in  $s$  or  $w$  or  $x$ , of rational number singular point, namely  $w = \pm 1/4, w = \pm 1/2, w = \pm 1, x = 1/16, x = 1, x = 1/4, x = 1/9, x = 1/25, x = 1/8, \dots$

interpretation is that, for some primes such that the golden number is ‘like’ a rational number<sup>67</sup>, an expression, like the first two ones in (C.4), can be seen as an algebraic one. Of course this is not true for almost all primes.

Similarly, the order four linear differential operator

$$(1 - x)^2 x^2 \cdot D_x^4 + 4 \cdot (1 - 2x)(1 - x) \cdot x \cdot D_x^3 + (1 - 14x + 14x^2) \cdot D_x^2 - 2 \cdot (1 - 2x) \cdot D_x \tag{C.5}$$

is also Fuchsian with solutions that can be expressed in terms of hypergeometric functions  ${}_3F_2$  and has quadratic number exponents as can be seen on the indicial polynomials corresponding respectively to  $x = 0, 1$ , coincide with (C.4) and for  $x = \infty$  with  $r^2(r - 1)^2$ . Again calculating the  $p$  curvature of this Fuchsian operator with the same rational Wronskian  $1/((x - 1)^4 \cdot x^4)$ , with algebraic *but not rational numbers* critical exponents (thus excluding global nilpotence) one finds, for a *subset of the prime numbers*, a nilpotent characteristic polynomial, namely  $T^4$  (with a minimal polynomial  $T^2$ ). Generically the characteristic polynomial rules out the nilpotence, since it reads

$$T^4 + \frac{p - 5}{x^{4p} + (p - 2)x^{3p} + x^{2p}} \cdot T^2. \tag{C.6}$$

Another simple example is the order-three linear differential operator

$$x^2 \cdot (1 - x) \cdot D_x^3 + x \cdot (2 - 3x) \cdot D_x^2 + (1 + 2x) \cdot D_x - 1, \tag{C.7}$$

which has exponents that can simply be expressed in terms of third root of unity and golden number, as can be seen on the indicial polynomial corresponding respectively to  $x = 0, 1, \infty$ :

$$r \cdot (r^2 - r + 1), \quad r \cdot (r - 1)^2, \quad r \cdot (r^2 - 3r + 1).$$

Again calculating the  $p$ -curvature of this Fuchsian operator with a rational Wronskian  $1/((x - 1) \cdot x^2)$  but *non-rational* critical exponents (thus excluding global nilpotence), one finds for a subset of the prime numbers ( $\dots, 109, 163, 181, 199, \dots$ ) a characteristic polynomial  $T^3$  with a minimal polynomial  $T^2$ , and for a smaller set of primes ( $\dots, 73, 271, \dots$ ) a characteristic polynomial  $T^3$  with a minimal polynomial  $T^3$ .

The miscellaneous examples we have displayed actually correspond to the generic situation of Fuchsian linear ODEs with *integer coefficients* (the proper framework we expect, at first sight, for lattice statistical mechanics quantities satisfying a linear ODE). However they are not globally nilpotent and thus are not ‘derived from geometry’: generically<sup>68</sup> a Fuchsian linear ODE does not have solutions that can be expressed as  $n$ -fold integrals of algebraic integrands.

#### Appendix D. Linear differential equation for $\tilde{\chi}_d^{(4)}(t)$

The linear differential operator for  $\tilde{\chi}_d^{(4)}(t)$  is of order eight, and has the *direct sum* decomposition

$$\begin{aligned} \mathcal{L}_8^{(4)} &= L_1^{(4)} \oplus L_3^{(4)} \oplus L_4^{(4)}, \quad \text{with } L_1^{(4)} = D_t + \frac{d}{dt} \ln \left( \frac{t - 1}{t} \right), \\ L_3^{(4)} &= D_t^3 + \frac{(5t^2 + 6t - 1)}{(1 + t)(t - 1)t} \cdot D_t^2 + \frac{(3t^3 + 6t^2 - 2t - 1)}{(1 + t)t^2(t - 1)^2} \cdot D_t - \frac{3}{2(1 + t)(t - 1)t^2}, \end{aligned} \tag{D.1}$$

<sup>67</sup> Namely the primes  $p$  such that  $r^2 - r - 1$  factorize in  $F_p$ . For instance,  $r^2 - r - 1$  factorizes into  $(r + 7)(r + 3) \pmod{11}$ ,  $(r + 4)(r + 14) \pmod{19}$ ,  $\dots$ ,  $(r + 58)(r + 252) \pmod{311}$ .

<sup>68</sup> In a mathematical perspective. In contrast, physics seems to favour the DFG framework for the minimal order ODEs.

where  $L_4^{(4)}$  is an order-four linear differential operator with apparent singularities  $t^2 - 10t + 1 = 0$ , that will not be displayed here. Introducing the order-one operator:

$$G_1 = D_t + \frac{d}{dt} \ln \left( \frac{(t^2 - 10t + 1)(t - 1)^6 \cdot t^4}{(t + 1)^8} \right) \tag{D.2}$$

one gets rid of these apparent singularities (desingularization), and obtains an order-five Fuchsian linear differential operator  $G_1 \cdot L_4^{(4)}$  which, after simple conjugations, can be simply written as:

$$\begin{aligned} t^4(t - 1)^3(t + 1)^2 \cdot G_1 \cdot L_4^{(4)} &= (t - 1)^3(t + 1)^2 t^4 \cdot D_t^5 \\ &+ (-9 + 11t^2 + 26t)(t - 1)^2(t + 1)t^3 \cdot D_t^4 \\ &+ (31t^4 + 172t^3 + 126t^2 - 140t + 19)(t - 1)t^2 \cdot D_t^3 \\ &+ 2(11t^5 + 107t^4 + 179t^3 - 271t^2 + 74t - 4)t \cdot D_t^2 \\ &+ (2t^4 + 43t^3 + 327t^2 - 199t + 19)t \cdot D_t + 3(t + 1)^3. \end{aligned} \tag{D.3}$$

The linear differential operator of order three,  $L_3^{(4)}$  is *actually equivalent* to the symmetric square  $\text{Sym}^2(L_E)$  of the second-order operator corresponding to the complete elliptic integral  $E(x^{1/2})$  (see (33)). This order-three linear differential operator  $L_3^{(4)}$  is therefore globally nilpotent. Actually, we have calculated its  $p$ -curvature of  $L_3^{(4)}$  and found that the corresponding characteristic polynomial (or minimal polynomial) reads  $T^3$ .

The order-four linear differential operator  $L_4^{(4)}$  is also globally nilpotent: we have calculated the  $p$ -curvature and found that the corresponding characteristic polynomial (or minimal polynomial) reads  $T^4$ . For the moment we have not been able to write one of his four solutions as a  ${}_4F_3$  hypergeometric function up to a pullback (trying to generalize subsection 6.2).

### Appendix E. Revisiting the global nilpotence of $\Phi_D^{(n)}$ for $n = 3, 4, 6$

#### E.1. Revisiting the global nilpotence of $\Phi_D^{(3)}$

The global nilpotence of  $\Phi_D^{(3)}$  can be understood from the factorization of the corresponding linear differential operator which can be seen as the direct sum of an operator of order three and of  $D_x$ :

$$\begin{aligned} D_x \oplus L_3, \quad \text{where} \quad L_3 &= z_2 \cdot L_1, \quad \text{with} \\ L_1 &= D_x + \frac{1}{2} \frac{d}{dx} \ln \left( \frac{(1 + 3x + 4x^2)(1 + 2x)(x - 1)}{(1 + x)^2} \right), \\ q_2 \cdot z_2 &= q_2 \cdot D_x^2 + 2 \cdot (1 + x) \cdot q_1 \cdot D_x + 4 \cdot q_0, \end{aligned} \tag{E.1}$$

where

$$\begin{aligned} q_2 &= x(1 - x)(1 + 4x)(1 + 2x)(1 - 4x)(1 + 3x + 4x^2)(1 + x)^2 \cdot Q_2, \\ Q_2 &= 3264x^8 + 56x^7 - 862x^6 + 3641x^5 + 1873x^4 + 149x^3 - 23x^2 + 2, \\ q_1 &= 1253\,376x^{15} + 1330\,688x^{14} - 492\,800x^{13} + 1432\,064x^{12} + 3680\,288x^{11} \\ &\quad + 1249\,562x^{10} - 1192\,677x^9 - 1051\,887x^8 - 317\,269x^7 \\ &\quad - 47\,698x^6 - 8120x^5 - 2801x^4 - 693x^3 - 50x^2 + 15x + 2, \\ q_0 &= 626\,688x^{15} + 1237\,248x^{14} + 237\,504x^{13} + 898\,720x^{12} + 3726\,900x^{11} \\ &\quad + 3657\,589x^{10} + 1424\,484x^9 + 315\,618x^8 + 122\,103x^7 \\ &\quad + 24\,147x^6 - 21\,786x^5 - 14\,389x^4 - 3444x^3 - 375x^2 - 9x + 2. \end{aligned}$$

The order-two linear differential operator  $z_2$  can be seen to be homomorph to  $Q_E$ , defined in (31), corresponding to the complete elliptic integral  $E(4x)$ :

$$z_2 \cdot W_1 = W_2 \cdot Q_E, \tag{E.2}$$

where  $W_1$  and  $W_2$  are two linear operators of order one. From the explicit expression of  $W_1$  one easily finds the following solution for  $z_2$ :

$$\frac{(12x^3 + 7x^2 + x - 2) \cdot E(4x)}{(1 + 3x + 4x^2)(1 - 4x)(1 + 2x)(1 - x)x} - \frac{(34x^4 + 11x^3 + 6x^2 + 7x + 2) \cdot K(4x)}{(1 + 2x)(1 + x)(1 + 3x + 4x^2)(1 - x)x}.$$

*E.2. Revisiting the global nilpotence of  $\Phi_D^{(4)}$*

The global nilpotence of  $\Phi_D^{(4)}$  can be understood from the factorization of the corresponding linear differential operator which can be seen as the direct sum of a linear operator of order three and of  $D_x$ :

$$D_x \oplus L_3, \quad \text{where} \quad L_3 = L_2 \cdot M_1, \quad \text{with}$$

$$M_1 = D_x + \frac{1}{2} \frac{d}{dx} \ln \left( \frac{4x - 1}{(x - 1)^2} \right),$$

$$L_2 = q_2 \cdot D_x^2 + (x - 1) \cdot q_1 \cdot D_x + q_0, \quad \text{where}$$

$$q_2 = (1 - 16x)(1 - 4x)(1024x^3 + 28x^2 - 42x + 1)(1 - x)^2 \cdot x,$$

$$q_1 = 262\,144x^6 - 228\,608x^5 - 4496x^4 + 19\,420x^3 - 3088x^2 + 125x - 2,$$

$$q_0 = 147\,456x^6 - 242\,624x^5 + 13\,376x^4 + 49\,864x^3 - 14\,530x^2 + 961x + 2. \tag{E.3}$$

The order-two linear differential operator

$$\tilde{Q}_E = D_x^2 + \frac{D_x}{x} + \frac{4}{(1 - 16x)x} \tag{E.4}$$

corresponding to the complete elliptic integral  $E(4\sqrt{x})$ , is equivalent to the linear differential operator  $L_2$

$$\tilde{Q}_E \cdot z_1 = s_1 \cdot L_2, \tag{E.5}$$

where  $z_1$  and  $s_1$  are two order-one linear differential operators.

*E.3. Revisiting the global nilpotence of  $\Phi_D^{(6)}$*

The linear differential operator for  $\Phi_D^{(6)}$  is an order-five Fuchsian linear operator which is the direct sum of  $D_x$  (here  $x = w^2$ ) and of an order-four operator which factorizes as a product of two order-two operators

$$L_{\Phi_D^{(6)}} = D_x \oplus L_4, \quad \text{where} \quad L_4 = M_2 \cdot L_2, \tag{E.6}$$

where  $M_2$  is a pretty large order-two linear differential operator (with a rational Wronskian) and

$$L_2 = D_x^2 - 2 \cdot (1 - 4x) \cdot \frac{P_1}{P_2} \cdot D_x - 2 \frac{P_0}{P_2}, \quad \text{with}$$

$$P_2 = (1 - 4x)^2(1 - x)(1 - 9x)(1 - 10x + 29x^2)$$

$$\times (1722x^6 - 3306x^5 + 2973x^4 - 1548x^3 + 403x^2 - 46x + 2),$$

$$\begin{aligned}
 P_1 &= 898\,884x^{10} - 2797\,104x^9 + 4902\,117x^8 - 5573\,337x^7 + 3999\,969x^6 \\
 &\quad - 1764\,005x^5 + 477\,136x^4 - 79\,113x^3 + 7883x^2 - 441x + 11, \\
 P_0 &= 898\,884x^{10} - 2559\,756x^9 + 3491\,100x^8 - 2205\,501x^7 + 556\,746x^6 \\
 &\quad + 92\,091x^5 - 92\,841x^4 + 23\,740x^3 - 3081x^2 + 226x - 8.
 \end{aligned}
 \tag{E.7}$$

The square of the Wronskian of  $L_2$  is a simple rational function. The Fuchsian linear operator  $L_2$  is such that the  $p$ -curvatures are zero for almost all primes, and therefore it has a basis of algebraic solutions. Note that the differential Galois group of  $L_2$  is isomorphic to the group of quaternions (eight elements). Its algebraic solutions correspond to an algebraic curve of genus  $g = 5$ . The equation of that algebraic curve reads

$$\begin{aligned}
 &21\,025 \cdot (2 - 36x + 218x^2 - 558x^3 + 553x^4 - 106x^5 + 27x^6)^4 \\
 &\quad - 58 \cdot p_2 \cdot (1 - 4x)^2(1 - x)^2(1 - 9x)^2(1 - 10x + 29x^2)^2 \cdot Z^2 \\
 &\quad + (1 - 4x)^4(1 - x)^4(1 - 9x)^4(1 - 10x + 29x^2)^4 \cdot Z^4 = 0,
 \end{aligned}
 \tag{E.8}$$

with

$$\begin{aligned}
 p_2 &= 1053x^{12} + 46\,836x^{11} - 429\,262x^{10} + 520\,760x^9 + 1315\,505x^8 - 3318\,300x^7 \\
 &\quad + 3056\,140x^6 - 1518\,520x^5 + 448\,000x^4 - 80\,280x^3 + 8552x^2 - 496x + 12.
 \end{aligned}$$

Again, these algebraic functions, roots of a genus 5 algebraic curve, can be expressed as linear combinations of complete elliptic integrals of the third kind with a ‘characteristic’ (first argument of the complete elliptic integral of the third kind) associated with a genus 3 curve (see appendix F.1).

We have also calculated the  $p$ -curvature of the (quite large) order-two Fuchsian linear differential operator  $M_2$  (for primes  $< 400$ ) and found that all these  $p$ -curvatures are nilpotent. One can actually prove that  $M_2$  is equivalent to the previous second-order operator (E.6), associated with  $E(4\sqrt{x})$ .

## Appendix F. Towards a geometrical interpretation of global nilpotence

### F.1. Towards an interpretation as periods of algebraic varieties: closed formula for $\Phi_D^{(n)}$

The integrals  $\Phi_D^{(n)}$  can all be expressed as sums of complete elliptic integrals of the third kind  $\Pi(y(w), w)$ , where the characteristic<sup>69</sup>  $y = y(w)$  corresponds to some algebraic curves:

$$\Phi_D^{(n)} = \sum_i A_i(w) \cdot \Pi(y_i(w), w), \quad P_n(y_i, w) = 0, \tag{F.1}$$

where  $A_i(w)$  are algebraic expressions of<sup>70</sup>  $w$  and  $P_n$  are simple polynomials of  $y_i$  and  $w$  with integer coefficients.

#### F.1.1. Towards an interpretation as periods of algebraic varieties: closed formula for $\Phi_D^{(3)}$ .

Let us give an exact expression for the integral  $\Phi_D^{(3)}$  (see (16)) in terms of complete elliptic integrals of the third kind<sup>71</sup>. Let us introduce  $f_1$  and  $f_2$  solutions of

$$\begin{aligned}
 &(1 - x - 4x^2)^2 \cdot (f_1^2 - 1) - 2x^2 \cdot (4x + 1)^2 \cdot (f_1 - 1) \\
 &\quad - 2(1 + 2x) \cdot (4x - 1) \cdot (f_1 + 1) = 0,
 \end{aligned}
 \tag{F.2}$$

<sup>69</sup> The first argument in a complete elliptic integral of the third kind is called the characteristic.

<sup>70</sup> In the following subsections, the calculations are expressed in terms of a variable  $x$  that is equal to  $w$  for  $n$  odd and to  $w^2$  for  $n$  even.

<sup>71</sup> We thank M Rybowicz for kindly providing us other closed expressions in terms of complete elliptic integrals of the third kind.

$$(1 + 3x + 4x^2) \cdot (f_2^2 - 1) + (1 + 2x)(1 - 4x) \cdot (f_2 + 1) - (1 + 4x)^2 \cdot (f_2 - 1) = 0, \tag{F.3}$$

which are *rational* curves that can be parametrized as follows:

$$x = \frac{4 + u^2}{2(8 - u^2)}, \quad f_1 = \frac{(u^2 + 6u - 8)^2}{(u^2 - 6u - 8)^2}, \quad f_2 = \frac{u^2 + 6u + 16}{u^2 - 6u + 16}.$$

Let us introduce the three *involutive* birational transformations:

$$J(f_i) = \frac{(1 + x) - (1 - x) \cdot f_i}{(1 - x) - (1 + x) \cdot f_i}, \quad H(f_i) = \frac{1}{f_i}, \quad I(f_i) = -f_i.$$

The non-involutive birational transformation  $I \circ J$  maps (F.2) onto (F.3), and, of course,  $J \circ I$  maps back (F.3) onto (F.2). These two rational curves are invariant under the (Hadamard) involution  $H: f_1 \rightarrow 1/f_1$  and  $f_2 \rightarrow 1/f_2$ . Note that  $H$  and  $I \circ J$  commute.

Let us introduce the expression

$$R(f, 4x) = \frac{2 \cdot \Pi(4xf, 4x)}{\pi} \cdot \sqrt{\frac{(4xf - 1)(4x - f)}{f}}, \tag{F.4}$$

one then easily finds that  $R(f, 4x)$  is such that

$$R(f, 4x) + R(1/f, 4x) = 1 + \frac{2}{\pi} \cdot K(4x) \cdot \sqrt{\frac{(4xf - 1)(4x - f)}{f}}, \tag{F.5}$$

$$\frac{d}{dx} R(f(x), 4x) = Q(x) \cdot \frac{2}{\pi} \cdot E(4x) + P(x) \cdot \frac{2}{\pi} \cdot K(4x),$$

for some  $Q(x)$  and  $P(x)$  that can be deduced from  $f = y(x)$ . These identities are valid for any  $y(x)$  and do not require  $y(x)$  to be, for instance, a rational function of  $x$ . The occurrence of several operators equivalent to  $L_E$  (associated with the complete elliptic integral  $E$ ) in the factorization of the linear differential operators corresponding to the  $\Phi_D^{(n)}$ , can be seen as a consequence of that identity (F.5) on the complete elliptic integral of the third kind for arbitrary characteristic  $y(x)$ .

Introducing a combination of four complete elliptic integrals of the third kind, with characteristics satisfying the rational curves (F.2), (F.3):

$$\Sigma = R(f_1, 4x) - R(1/f_1, 4x) + R(f_2, 4x) - R(1/f_2, 4x), \tag{F.6}$$

$\Phi^{(3)}$  can then be simply written in terms of  $\Sigma$  as

$$\Phi_D^{(3)} = \frac{1}{8} + \frac{1}{12\pi} \cdot \frac{(1 + 4x)(1 - 9x^2 - 12x^3)}{(1 - x - 4x^2)(1 + 3x + 4x^2)} \cdot K(4x) + C \cdot \Sigma,$$

where

$$C = -\frac{i}{48} \cdot \frac{(1 + x)}{\sqrt{1 + 2x}\sqrt{1 - x}\sqrt{1 + 3x + 4x^2}}. \tag{F.7}$$

*F.1.2. Towards an interpretation as periods of algebraic varieties: closed formula for  $\Phi_D^{(4)}$ .*

For  $\Phi_D^{(4)}$ , and in a similar way as for  $\Phi_D^{(3)}$ , one has now the two *rational* curves:

$$3 \cdot (1 + 4x)(f_1 + 1)^2 + (1 - 4x)(f_1 - 1)^2 = 0, \tag{F.8}$$

$$27 \cdot (1 + 4x)x^4(f_2 + 1)^2 + (1 - 4x)(2 - 5x^2)^2(f_2 - 1)^2 = 0. \tag{F.9}$$



A simple parametrization reads

$$x = \frac{1}{4} \cdot \frac{u^2 + 3}{u^2 - 3}, \quad f_1 = \frac{1 + u}{1 - u}, \quad f_2 = \left( \frac{1 - u}{1 + u} \right) \cdot \left( \frac{u^2 - 4u - 9}{u^2 + 4u - 9} \right)^2.$$

Let us introduce the three involutive birational transformations:

$$J(f_i) = \frac{(1 - x^2) - (1 - 4x^2) \cdot f_i}{(1 - 4x^2) - (1 - x^2) \cdot f_i}, \quad H(f_i) = \frac{1}{f_i}, \quad I(f_i) = -f_i.$$

the non-involutive birational transformation  $J \circ I$  maps the first curve (F.8) onto the second curve (F.9), and of course its inverse  $I \circ J$  maps back (F.9) onto (F.8). Note that these two rational curves are actually invariant under the (Hadamard) involution  $H: f_1 \rightarrow 1/f_1$  and  $f_2 \rightarrow 1/f_2$ . The rational curve (F.8) has, of course, many other involutive birational automorphisms:

$$(x, f_1) \rightarrow \left( -\frac{1 + 5x}{5 + 16x}, -f_1 \right).$$

Using the same form for  $\Sigma$  as in (F.6), the solution  $\Phi_D^{(4)}$  can now be written simply as

$$\Phi_D^{(4)} = \frac{1}{32} + \frac{1}{144\pi} \cdot E(4x) + \frac{8x^4 + 9x^3 - 11x^2 - 1}{72\pi(2x + 1)(x^2 + 3x - 1)} \cdot K(4x) + C \cdot \Sigma,$$

where

$$C = \frac{i\sqrt{3}}{864} \cdot \frac{x^2 - 1}{\sqrt{1 - 4x^2}}$$

*F.1.3. Towards an interpretation as periods of algebraic varieties: closed formula for  $\Phi_D^{(5)}, \Phi_D^{(6)}, \Phi_D^{(7)}$  and  $\Phi_D^{(8)}$*  For higher values of  $n$  ( $n \geq 5$ ) the algebraic curves  $P_n(y_i, w) = 0$ , which occur in the closed formula (F.1) for the  $\Phi_D^{(n)}$  are no longer genus zero curves but higher genus curves. For instance for the  $\Phi_D^{(5)}$  and  $\Phi_D^{(6)}$ , these curves are genus 3 curves. In terms of  $q = 4wy$  they read respectively:

$$q^4 - 4q^3 - 4(4w^2 - 3) \cdot q^2 + 16(2w^2 - 1) \cdot q + 8(w + 1)(4w^3 - 3w^2 - w + 1) = 0$$

and

$$10q^4 - 40q^3 - 5(3w^2 - 2) \cdot q^2 + 80(3w^2 - 1) \cdot q + 32(w - 1)(2w + 1)(2w - 1)(w + 1) = 0.$$

For  $\Phi_D^{(7)}$  and  $\Phi_D^{(8)}$  these curves are genus 10 curves.

For  $\Phi_D^{(7)}$ , in terms of  $q = 4wy$ , one gets

$$q^6 - 6q^5 - 6(-5 + 4w^2)q^4 + 16(6w^2 - 5) \cdot q^3 + 24(6w^4 + 5 - 12w^2)q^2 - 96(w - 1)(w + 1)(3w^2 - 1) \cdot q + 32 - 128w^6 - 192w^2 + 288w^4 + 32w^5 = 0.$$

The vanishing conditions  $\delta_n = 0$  of the discriminants in  $q$  of these polynomials, associated with  $\Phi_D^{(n)}$ , read respectively as follows:

$$\delta_5 = (1 - 2w)(1 + 4w)(1 - 2w - 4w^2)(w - 1)^2(1 + w - 3w^2 - 4w^3)^2 = 0,$$

$$\delta_6 = (1 + 4w)(1 + 2w)(1 - 2w)(1 - 4w)(1 - 10w^2 + 29w^4)^2 = 0,$$

$$\delta_7 = (1 + 4w)(1 - 2w - 4w^2)(1 - 2w - 8w^2 + 8w^3)(1 - w)^2$$

$$(w^3 - w^2 - 2w + 1)^2 \cdot d_7^2 = 0$$

$$d_7 = 1 + 3w - 10w^2 - 35w^3 + 5w^4 + 62w^5 + 17w^6 - 32w^7 - 16w^8.$$

With the exception of  $(1 + 4w)(1 - 4w) = 0$  in  $\delta_6 = 0$ , these polynomial conditions are, respectively, the singularities of the linear ODEs for  $\Phi_D^{(5)}$ ,  $\Phi_D^{(6)}$  and  $\Phi_D^{(7)}$ , where  $w$  has been changed into  $(-w)$ .

**Appendix G. Atkin’s modular curves and Weber’s modular functions**

The classical modular curve [127] which corresponds to the duplication of the ratio of periods of the elliptic curves [128]  $j = j(\tau) \rightarrow j' = j(2\tau)$ :

$$j^2 \cdot j'^2 - (j + j') \cdot (j^2 + 1487 \cdot jj' + j'^2) + 3 \cdot 15^3 \cdot (16j^2 - 4027jj' + 16j'^2) - 12 \cdot 30^6 \cdot (j + j') + 8 \cdot 30^9 = 0, \tag{G.1}$$

of course symmetric by  $j \leftrightarrow j'$ , is well known to be a genus zero curve with the rational parametrization:

$$j = j_2(z) = \frac{(z + 16)^3}{z}, \quad j' = \frac{(z + 256)^3}{z^2} = j_2\left(\frac{2^{12}}{z}\right). \tag{G.2}$$

This is the duplication formula of Klein’s absolute invariant. The involution  $z \rightarrow 2^{12}/z$  is the Atkin involution [88]. Recall that the rational variable  $z$  can simply be expressed in term of the Dedekind eta function:

$$z = 2^6 \cdot \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24}. \tag{G.3}$$

The modular curve (G.1) can also be parametrized as

$$j = 256 \frac{(1 - k^2 + k^4)^3}{(1 - k^2)^2 k^4}, \quad j' = 16 \frac{(1 + 14k^2 + k^4)^3}{(1 - k^2)^4 k^2} = j\left(\frac{2\sqrt{k}}{1+k}\right),$$

where the occurrence of the Landen transformation becomes explicit, or:

$$j = \frac{(1 - 16w^2 + 16w^4)^3}{(1 - 16w^2)w^8}, \quad j' = \frac{(256w^4 - 16w^2 + 1)^3}{(1 + 4w)^2(1 - 4w)^2w^4}.$$

Similarly, the triplication formula of Klein’s absolute invariant can also be written rationally:

$$j = j_3(z) = \frac{(z + 27)(z + 3)^3}{z}, \quad j' = \frac{(z + 27)(z + 243)^3}{z^3} = j_3\left(\frac{3^6}{z}\right),$$

where  $z \rightarrow 3^6/z$  is, again, the Atkin involution. The elimination of the rational variable  $z$  yields the classical modular curve which corresponds to the triplication of the ratio of periods of the elliptic curves  $j = j(\tau) \rightarrow j' = j(3\tau)$ :

$$j^4 + j'^4 - j^3 \cdot j'^3 + 2232 \cdot j'^2 j^2 \cdot (j + j') - 1069956 \cdot (jj'^3 + j'j^3) + 2587918086j^2j'^2 + 36864000 \cdot (j + j') \cdot (j^2 + 241433jj' + j'^2) + 16777216000000 \cdot (27j^2 + 27j'^2 - 45946jj') + 185542587187200000000 \cdot (j + j') = 0. \tag{G.4}$$

The genus zero classical modular curve (G.4) is, of course, symmetric by  $j \leftrightarrow j'$ . The rational variable  $z$  can simply be expressed in term of the Dedekind eta function [129, 130]:

$$z = 3^s \cdot \left(\frac{\eta(3\tau)}{\eta(\tau)}\right)^{2s}, \quad s = 6. \tag{G.5}$$

We will not write (though it is straightforward) the classical modular curve (50) which corresponds to  $j = j(\tau) \rightarrow j' = j(6\tau)$ . This classical modular curve is again a genus zero

curve, corresponding to a polynomial relation symmetric by  $j \leftrightarrow j'$  with integer coefficients and it can simply be obtained from the elimination of  $z$  in the rational parametrization (51) with, again, an Atkin involution  $z \rightarrow 2^3 \cdot 3^2/z$ .

Note that the rational functions occurring in the rational parametrization of all these genus zero classical modular curves are related together by some rational change of variables:

$$j_6(z) = j_2\left(\frac{z \cdot (z+8)^3}{z+9}\right) = j_3\left(\frac{z \cdot (z+9)^2}{z+8}\right). \quad (\text{G.6})$$

The integers  $N$  such that the modular curves,  $P_N(j(\tau), j(N\tau)) = 0$ , are genus zero is a highly selected set of integers corresponding to the Monstrous Moonshine phenomenon [131, 132].

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